Combinatorial conditions for low rank solutions in semidefinite programming

Varvitsiotis, A.

Publication date:
2013

Document Version
Publisher's PDF, also known as Version of record

Combinatorial Conditions
for
Low Rank Solutions
in
Semidefinite Programming

Antonios Varvitsiotis

This research has been carried out at the Centrum Wiskunde & Informatica in Amsterdam.

ISBN: 978-90-5668-371-9

Combinatorial Conditions
for
Low Rank Solutions
in
Semidefinite Programming

Proefschrift

ter verkrijging van de graad van doctor aan
Tilburg University
op gezag van de rector magnificus
prof. dr. Ph. Eijlander
in het openbaar te verdedigen ten overstaan van een
door het college voor promoties aangewezen commissie
in de aula van de Universiteit
op maandag 25 november 2013
om 16.15 uur

door

Antonios Varvitsiotis

geboren op 13 september 1982
te Athene, Griekenland

Promotor:	prof. dr. M. Laurent
Overige leden:	prof. dr. E. de Klerk
	prof. dr. W. H. Haemers
	prof. dr. C. Helmberg
	prof. dr. A. Schrijver
	prof. dr. F. Vallentin
	prof. dr. ir. E. R. van Dam

Acknowledgements

This thesis is the outcome of my 4-year stay as a graduate student at Centrum Wiskunde & Informatica in Amsterdam. Working towards the completion of this thesis has been an extremely challenging task. Nevertheless, I have been fortunate enough to be surrounded by people that, in a variety if ways, have contributed to its completion. I would now like to take the opportunity to thank all of you.

First of all, I would like to express my gratitude to my supervisor Monique Laurent. Working under Monique's supervision has been an invaluable experience, both on a scientific but also on a personal level. I greatly admire Monique's energy and limitless enthusiasm for mathematical research which has inspired me throughout all these years. Most importantly, Monique taught me to only settle for the highest of standards for my work, and this is a lesson I will cherish forever.

I would also like to thank the members of my Phd committee: prof. dr. E. de Klerk, prof. dr. W. H. Haemers, prof. dr. C. Helmberg, prof. dr. A. Schrijver, prof. dr. F. Vallentin, and prof. dr. ir. E.R. van Dam for giving me the honor of participating in my Phd committee. Their comments and their suggestions were extremely helpful and helped me to significantly improve the correctness, the structure and the overall presentation of the thesis.

Furthermore, I would like to thank my colleagues (past and present) from the Networks and Optimization group at CWI: Krzysztof Apt, Dion Gijswijt, Ross Kang, Anargyros Katsampekis, Bart de Keijzer, Teresa Piovesan, Guus Regts, Guido Schäfer, Lex Schrijver, Tobias Müller, Frank Vallentin, Gauthier van den Hove and Fernando Mário de Oliveira Filho. It has been a privilege to be surrounded by such great people that are responsible for creating a welcoming and hospitable working environment at CWI. Moreover, I would like to thank our secretary, Susanne van Dam, for always being extremely helpful and Irma van Lunenburg from the P&O department for helping me with various administrative issues.

The last four years I was blessed with many wonderful friends that made my stay in the Netherlands an unforgettable experience. Orestis Telelis was at CWI during the first year of my stay in Amsterdam and we had some wonderful times together. I am privileged to count him as a dear friend. During the third year of my stay at CWI, I was fortunate enough that Marianna Eisenberg-Nagy joined our group. Throughout that year I closely collaborated with Marianna and I was amazed by her scientific brilliance. During my time at CWI, I switched offices many times, the only fixed point being my officemate Guus Regts. Together with Guus we had countless discussions, sharing our hopes and worries for the future and I consider him to be a great friend. Moreover, I would like to thank Giannicola Scarpa, with whom we shared a flat in Diemen for two years, for being a wonderful flatmate and a great friend. Lastly, I would also like to thank Alp Bassa, Jop Briët, Nacho Cascudo, Evan de Corte and Fernando de Melo for all the fun times and some wonderful memories.

I also wish to express my gratitude to my Greek friends that have been there for me throughout this period: Ilias Koutsamplelas, Nicki Panayi and Markos Panikidis. My deepest gratitude goes to Afroditi Vergopoulou for being the loving, caring and supporting person she is.

Lastly, but most importantly, I would like to thank my family, my father Efstratios and my sister Antonia, for their unconditional support. Without them this thesis would have never come into existence. Especially, I want to dedicate this thesis to my mother, who recently passed away. My mother was always very supportive of my dream to pursue a Phd and I know she would have been proud now that this work is finally completed.

Amsterdam, October 2013

Contents

1
Introduction

1.1 Motivation for this thesis

A semidefinite program (SDP) is a convex program defined as the minimization of a linear function over an affine section of the cone of positive semidefinite (psd) matrices. The central theme of this thesis is *the search for combinatorial conditions guaranteeing the existence of low-rank optimal solutions to semidefinite programs*. Throughout this thesis we focus on combinatorial conditions that are expressed in terms of the sparsity pattern of the coefficient matrices in the semidefinite program.

Results ensuring the existence of low-rank optimal solutions to SDP's are important for approximation algorithms. Indeed, semidefinite programs are widely used as convex tractable relaxations for hard combinatorial problems. Then, the rank one solutions typically correspond to the desired optimal solutions of the initial discrete problem and low rank solutions can decrease the error of the rounding methods and lead to improved performance guarantees.

Low-rank solutions to SDP's are also relevant to the study of geometric representations of graphs. In this setting we consider representations obtained by assigning vectors to the vertices of a graph, where we impose restrictions on the vectors labeling adjacent vertices (e.g. orthogonality, unit distance). Then, questions related to the existence of low-dimensional representations can be reformulated as the problem of deciding the existence of a low rank solution to an appropriate semidefinite program, and are connected to interesting graph properties.

The problem of identifying combinatorial conditions for the existence of low-rank solutions to SPD's was raised by Lovász in [86]. Quoting Lovász [86, Problem 8.1] it is important to "find combinatorial conditions that guarantee that the semidefinite relaxation has a solution of rank 1". Furthermore, the version of this problem "with low rank instead of rank 1, also seems very interesting".

In the following sections we introduce semidefinite programming and motivate its relevance to the fields of approximation algorithms and geometric graph representations. In both cases, our main objective is to illustrate the fact that, the underlying combinatorial information of a semidefinite program can provide guarantees for the existence of low-rank optimal solutions.

1.1.1 Semidefinite programming

A semidefinite program in canonical primal form looks as follows:

$$\begin{aligned} \inf\ & \langle A_0, X\rangle \\ \text{subject to } & \langle A_k, X\rangle = b_k, \qquad k = 1, \ldots, m \\ & X \succeq 0, \end{aligned} \tag{1.1}$$

where C, A_k $(0 \leq k \leq m)$ are n-by-n symmetric matrices and $\langle \cdot, \cdot \rangle$ denotes the usual Frobenius inner product of matrices. The generalized inequality $X \succeq 0$ means that X is positive semidefinite, i.e., all its eigenvalues are nonnegative. The matrices C, A_k $(0 \leq k \leq m)$ are the *coefficient* matrices of the semidefinite program.

Semidefinite programming is a far reaching generalization of linear programing with a wide range of applications in a number of disparate areas such as approximation algorithms [92], control theory [97], polynomial optimization [79] and quantum information theory [30].

The field of semidefinite programming has grown enormously in recent years and this success can be attributed to the fact that SDP's have significant modeling power, exhibit a powerful duality theory and there exist efficient algorithms, both in theory and in practice, for solving them. Starting with the seminal work of Goemans and Williamson on the MAX CUT problem, SDP's have proven to be an invaluable tool in the design of approximation algorithms for hard combinatorial optimization problems. This success is vividly illustrated by the fact that currently, many SDP based approximation algorithms are essentially optimal for a number of problems, assuming the validity of the Unique Games Conjecture (e.g. [65, 91]).

The first landmark application of semidefinite programming is the work of Lovász on approximating the Shannon capacity of graphs [87]. The Lovász ϑ-function is defined as the optimal value of the following semidefinite program:

$$\max \sum_{i,j=1}^{n} X_{ij} \ \text{ s.t. } \ \sum_{i=1}^{n} X_{ii} = 1,\ X_{ij} = 0\ (ij \in E),\ X \succeq 0. \tag{1.2}$$

The Lovász ϑ-function was introduced in [87] as an efficiently computable upper bound to the Shannon capacity of a graph. Additionally, the celebrated *sandwich theorem* by Lovász states that

$$\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G),$$

where $\omega(G)$ denotes the *clique number* and $\chi(G)$ the *chromatic number* of the graph. Thus $\vartheta(\overline{G})$ is an efficiently computable approximation to both the clique number and the chromatic number of graph and currently gives rise to the only known polynomial time algorithm for calculating these parameters in perfect graphs.

The problem of identifying low-rank solutions to general semidefinite programs has received a significant amount of attention (e.g. [24, 86] and further references therein). The most important result in this direction, which has been rediscovered in many different versions ([22, 98]), is that if a semidefinite program of the form (1.1) is feasible then it also has a feasible solution of rank at most

$$\left\lfloor \frac{\sqrt{8m+1}-1}{2} \right\rfloor, \tag{1.3}$$

and moreover this bound is in general the best possible (cf. Section 3.2) [49, 23].

Nevertheless, the bound given in (1.3) is valid for arbitrary SDP's and it does not take into consideration the sparsity of the coefficient matrices. To encode this information, with any semidefinite program (P) of the form (1.1) we associate a graph $\mathscr{A}_P = (V_P, E_P)$, called the *aggregate sparsity pattern* of (P), where $V_P = \{1, \dots, n\}$ and $ij \in E_P$ if and only if there exists $k \in \{0, 1, \dots, m\}$ such that $(\mathscr{A}_k)_{ij} \neq 0$.

In the following sections we will see that the combinatorial properties of the aggregate sparsity pattern can lead to guarantees for the existence of low-rank optimal solutions to semidefinite programs.

1.1.2 Approximation algorithms

Most optimization problems of practical interest are known to be NP-hard, which means that unless P = NP, these problems are hard to solve to optimality, in the worst case. However, in contrast to this worst case pessimism, for practical purposes it is usually sufficient to settle for near-optimal solutions that can be attained in polynomial time. This stimulated the development of the field of approximation algorithms where the goal is to obtain provably, near-optimal solutions in polynomial time. An algorithm is called a *ρ-approximation algorithm* ($\rho \leq 1$) for a maximization problem if for every instance of the problem the algorithm returns a solution whose value is at least ρ times the value of the optimal solution.

Although most physical systems are inherently nonlinear, approximation algorithms based on linear programming relaxations have proven to be an extremely powerful tool for addressing a wide range of hard optimization problems of significant practical interest. The idea underlying this algorithmic paradigm is to model combinatorial problems as integer programs, solve the corresponding linear programming relaxation, round the optimal fractional solution to a feasible solution for the original problem and then compare the value of the rounded solution to the value of the optimal fractional solution. This results in an approximation algorithm whose approximation ratio cannot exceed the *integrality gap* of this relaxation, i.e., the infimum of the ratio between the original problem and its relaxation.

While this approach has proven to be successful for a wide range of combinatorial optimization problems, there are some notable exceptions that do not succumb to purely linear methods. A prominent example is the MAX CUT problem, one of the most extensively studied combinatorial optimization problems. A *cut* in a graph is a partition of the vertices of the graph into two disjoint subsets and the corresponding *cut-set* consists of the edges whose endpoints belong to different sets of the partition. In an edge-weighted graph, the *weight* of a cut is defined as the sum of the weights of the edges that cross the cut. In the MAX CUT problem we are given an edge-weighted graph and the goal is to find a cut of maximum weight.

The (decision version of the) MAX CUT problem is one the first problems that were proven to be NP-complete [63]. In terms of its approximability properties, it is well known that MAX CUT cannot be approximated in polynomial time within any constant factor larger than $16/17 \approx 0.941$ [59]. On the other hand, there is a trivial randomized 1/2-approximation algorithm obtained by placing each vertex on either side of the partition independently with probability 1/2, and this approximation ratio is matched by the trivial greedy algorithm.

Despite significant efforts, all linear programing approaches for the MAX CUT problem failed to improve upon the 1/2 factor, which was the state of the art for decades [108]. The same fate was also in store for the powerful LP-hierarchies, that provide a systematic way for constructing hierarchies of relaxations that converge to the cut polytope [40, 119].

A major breakthrough took place in 1995 when Goemans and Williamson devised an 0.878-approximation algorithm for MAX CUT (with nonnegative edge weights), which was based on semidefinite programming [51]. Recently, it was shown by Khot et al. that assuming the validity of the Unique Games Conjecture, this approximation ratio is the best possible for MAX CUT [65].

Nevertheless, under the assumption that the Goemans and Williamson SDP relaxation has a low-rank optimal solution, it is possible to devise a more sophisticated rounding scheme, and as a result, one can improve slightly on the 0.878 approximation ratio. Specifically, assuming that the SDP relaxation has an optimal solution of rank 2 (resp. 3) this leads to a $\frac{32}{25+5\sqrt{5}} \approx 0.884458$ approximation ratio (resp. 0.8818) [14]. This fact illustrates vividly that results guaranteeing the existence of low-rank optimal solutions to semidefinite programs can be of great importance for approximation algorithms.

1.1.3 Geometric representations of graphs

In this section we consider Euclidean representations of graphs and show that various questions concerning the existence of such representations can be formulated as the problem of identifying low-rank solutions to a certain semidefinite program. Furthermore, we will see that taking into consideration the structure of the underlying graph can lead to significant improvements upon the general bound given in (1.3). Our exposition of this material follows closely [24, §2.15].

In this setting we are given as input a graph $G = (V = [n], E)$ and a vector $w = (w_{ij}) \in \mathbb{R}^E_+$ which corresponds to an assignment of nonnegative edge weights to its edges. An edge-weighted graph (G, w) is called *d-realizable* if there exist an assignment of vectors $p_1, \ldots, p_n \in \mathbb{R}^d$ to the vertices of the graph G such that

$$\|p_i - p_j\| = w_{ij} \text{ for all } ij \in E,$$

where $\|\cdot\|$ denotes the Euclidean norm. Given an edge-weighted graph (G, w), the *realization problem* asks whether there exists some integer $d \geq 1$ for which (G, w) is d-realizable. In the Distance Geometry community this problem is known as the *Euclidean distance matrix completion problem* [71]. Similarly, the *d-realization problem* asks wether an edge-weighted graph (G, w) is d-realizable. Moreover, assuming that an edge-weighted graph (G, w) is realizable (in some dimension) one can ask for the smallest dimension where a realization is possible.

All of these problems have received significant attention due to their relevance to molecular conformation problems in chemistry [37] and multidimensional scaling in statistics [41], among others. As we will now see, these problems can be phrased in the language of semidefinite programming.

Before we give the details of this reformulation we introduce some terminology. For a family of vectors $p_1, \ldots, p_n$, their *Gram matrix*, denoted by $\mathrm{Gram}(p_1, \ldots, p_n)$, is the n-by-n matrix whose (i, j) entry is given by $p_i^\mathsf{T} p_j$. The Gram matrix of any family of vectors is positive semidefinite since, for every $x \in \mathbb{R}^n$, we have $x^\mathsf{T}\mathrm{Gram}(p_1, \ldots, p_n)x = \|\sum_{i=1}^n x_i p_i\|^2 \geq 0$. Conversely, it is well-known that every psd matrix is the Gram matrix of some family of vectors; cf. Theorem 2.3.1. Setting $X = \mathrm{Gram}(p_1, \ldots, p_n)$, we have that the rank of X is equal to the dimension of the linear span of the vectors $p_1, \ldots, p_n$. Moreover, we have that

$$\|p_i - p_j\|^2 = \|p_i\|^2 + \|p_j\|^2 - 2p_i^\mathsf{T} p_j = \langle (e_i - e_j)(e_i - e_j)^\mathsf{T}, X\rangle,$$

where by e_i $(i \in [n])$ we denote the standard basis vectors in $\mathbb{R}^n$.

Following these observations it is clear that deciding whether a weighted graph (G, w) is realizable is equivalent to deciding the feasibility of the following SDP

$$\langle (e_i - e_j)(e_i - e_j)^\mathsf{T}, X\rangle = w_{ij} \text{ for all } ij \in E, \ X \text{ positive semidefinite.} \tag{1.4}$$

Furthermore, the d-realizability problem amounts to deciding the feasibility of the semidefinite program (1.4), augmented with the additional constraint

$$\mathrm{rank} X \leq d.$$

Specializing the general bound (1.3) to the semidefinite program (1.4) it follows that, if a weighted graph (G, w) is realizable then it also admits a realization in $\left\lfloor \frac{\sqrt{8|E|+1}-1}{2} \right\rfloor$-dimensional Euclidean space. As we will now see, by taking into consideration the structure of the graph G, this bound can be significantly improved.

The crucial ingredient in this approach is the notion of the *Euclidean dimension* of a graph, denoted by $\mathrm{ed}(\cdot)$, which was introduced in [25, 26]. The Euclidean dimension of a graph $G = ([n], E)$ is defined as the minimum integer $d \geq 1$ with the following property: For any family of vectors $p_1, \ldots, p_n$ there exist vectors $q_1, \ldots, q_n \in \mathbb{R}^d$ such that

$$\|p_i - p_j\| = \|q_i - q_j\| \ \text{ for all } \ ij \in E.$$

Equivalently, the Euclidean dimension of a graph can be expressed as the smallest integer $d \geq 1$ with the following property: For every $w \in \mathbb{R}_+^E$ for which (1.4) is feasible, it also has a feasible solution of rank at most d. Notice that this quantity is well defined, since whenever (1.4) is feasible, it has a solution of rank at most n.

In [25, 26] it is shown that the parameter $\mathrm{ed}(\cdot)$ is minor monotone, i.e., if H is a minor of G then $\mathrm{ed}(H) \leq \mathrm{ed}(G)$. Recall that a graph H is a *minor* of a graph G, if H can be obtained from G by a series of edge deletions, edge contractions and isolated node deletions, ignoring any loops or multiple edges that may arise in the process. By the celebrated *graph minor theorem* of Robertson and Seymour [115], it follows that for any fixed integer $k \geq 1$, the graphs satisfying $\mathrm{ed}(G) \leq k$ can be characterized by a finite list of minimal forbidden minors. In [25, 26] the graphs with small Euclidean dimension are characterized:

- $\mathrm{ed}(G) \leq 1$ if and only if G has no K_3-minor,
- $\mathrm{ed}(G) \leq 2$ if and only if G has no K_4-minor,
- $\mathrm{ed}(G) \leq 3$ if and only if G has no K_5 and $K_{2,2,2}$-minors.

Here, $K_{2,2,2}$ denotes the octahedral graph; cf. Figure 5.1. These bounds yield significant improvements upon the general bound (1.3) and the example of the graph realizability problem strongly highlights the fact that combinatorial information present in the coefficient matrices of a semidefinite program can be translated into guarantees for the existence of low rank solutions.

1.2 Contributions of this thesis

In this section we introduce and give some relevant background material concerning the two problems whose study forms the main body of this thesis. Along the way we also highlight the main contributions of this thesis and additionally we collect the most important problems that remain open.

1.2.1 The positive semidefinite matrix completion problem

The first problem that figures prominently throughout this thesis is the positive semidefinite (psd) matrix completion problem. Before we give the precise statement of this problem we first introduce some necessary definitions. A *partial matrix* is a "matrix" where only a subset of its entries is specified. Throughout this thesis we assume that the diagonal entries are always specified and moreover that the partial matrix is symmetric in the following sense: if the entry (i,j) is specified, the same holds for entry (j,i) and moreover they have the same value. As a running example in this section consider the partial matrix

$$\begin{pmatrix} 1 & 1 & ? & 0 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ 0 & ? & 1 & 1 \end{pmatrix}, \tag{1.5}$$

where the unspecified entries are indicated by the question marks.

The *support graph* of an n-by-n partial matrix A is the graph on n nodes where i and j are adjacent if and only if the entry A_{ij} is specified. If G is the support graph of a partial matrix A, we also say that A is a *G-partial matrix*. As an example, the support graph of the partial matrix given in (1.5) is the cycle of length 4 and thus it is a C_4-partial matrix. A *G-partial psd matrix* is a partial matrix for which every fully specified principal submatrix is positive semidefinite. For example, the partial matrix given in (1.5) is a C_4-partial psd matrix. Indeed, the only fully specified principal submatrices are $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and they are both psd. For a graph $G=(V,E)$ it will be convenient to collect all the entries of a G-partial matrix in a vector $a \in \mathbb{R}^{V\cup E}$, and this is a convention that we follow throughout this thesis.

In the *positive semidefinite matrix completion problem* we are given as input a G-partial matrix and the question is to decide whether there exists an assignment of real values to the unspecified entries such that the resulting matrix is psd. Formally, given a graph $G=(V=[n],E)$ and a vector $a \in \mathbb{R}^{V\cup E}$ we want to decide whether there exists a real symmetric n-by-n matrix X satisfying

$$X_{ij} = a_{ij} \ \text{ for all } \{i,j\} \in V \cup E, \ \text{ and } \ X \succeq 0. \tag{1.6}$$

A partial matrix $a \in \mathbb{R}^{V\cup E}$ for which (1.6) is feasible is called *completable*. Notice that the relation $X_{ij} = a_{ij}$ can be expressed as $\langle (e_i e_j^{\mathsf{T}} + e_j e_i^{\mathsf{T}})/2, X\rangle = a_{ij}$, where by e_i ($i \in [n]$) we denote the standard basis vectors in $\mathbb{R}^n$. This shows that the psd matrix completion problem is a semidefinite programming feasibility problem.

The psd matrix completion problem has received significant attention in the literature (see e.g. [75] and further references therein). An important special case is the completion problem for correlation matrices (psd matrices whose diagonal elements are equal to one) which plays an important role in the study of statistical and stochastic models. Indeed, in many practical applications, and especially in high dimensional models, it is a frequently occurring phenomenon that only a subset of the set of pairwise correlations is known, due to data limitations or the dynamic nature of the underlying problem. In such a scenario it is desirable to determine a psd completion of the partial covariance matrix in order to be able to use the statistical or stochastic model at hand. For further background concerning correlation matrices the reader is referred to Section 3.2.2.

Given a partial matrix $a \in \mathbb{R}^{V \cup E}$ for which (1.6) is feasible, one can also ask for the smallest rank of a feasible solution of (1.6), and the algorithmic question of finding an (approximate) solution (of smallest rank). Among all psd completions of a partial matrix, the ones with the lowest possible rank are of particular importance. Indeed, the rank of a matrix is often a good measure of the complexity of the data it represents. This is illustrated by the fact that abstract notions like complexity and dimensionality can be often expressed by means of the rank of an appropriate matrix.

As an example, it is well known that the minimum dimension of a Euclidean embedding of a finite metric space can be expressed as the rank of an appropriate positive semidefinite matrix (cf. Section 6.1) and in applications, one is often interested in embeddings in low dimension, say 2 or 3.

The problem of computing (approximate) low rank psd (or Euclidean) completions of a partial matrix is a challenging non-continuous, non-convex problem which, due to its great importance, has been extensively studied (see, e.g., [16, 5, 112], the recent survey [71] and further references therein).

However, it is known that by looking at the structure of the support graph it is possible to identify tractable instances. For instance, when the support graph is chordal (i.e., has no induced circuit of length at least 4), all the questions posed above have been fully answered. Clearly, if a G-partial matrix is completable then it also is G-partial psd. It turns out that this condition is also sufficient when the support graph is chordal. Formally we have that if G is chordal then, a G-partial matrix is completable if and only if it is G-partial psd [55]. In fact, this is a characterization of chordal graphs in the following sense: If the graph G is not chordal then there exist G-partial psd matrices which are *not* completable. The C_4-partial matrix given in (1.5) is such an example. Indeed, as we have already argued, all its completely specified submatrices are psd but, as it is easy to verify, this partial matrix does not admit a completion to a full psd matrix. Furthermore, for a chordal graph G, given a G-partial psd matrix $a \in \mathbb{Q}^{V \cup E}$ it is known that there exists a *rational* psd completion and this can be calculated in polynomial time (in the bit model of computation) [76]. Lastly, the minimum possible rank of such a completion is equal to the largest rank of the fully specified principal submatrices.

Further combinatorial characterizations (and some efficient algorithms for completions – in the real number model) exist for graphs with no K_4-minor and more generally when excluding certain splittings of wheels [21, 74, 76].

Our contributions

A central problem in this thesis is to understand how to use the combinatorial structure of the support graph, to show the existence of low-rank feasible solutions to (1.6). With this in mind, in Chapter 5 we introduce a new graph parameter, called the *Gram dimension of a graph*, which we denote by $\mathrm{gd}(\cdot)$. The Gram dimension of a graph is defined as the smallest integer $k \geq 1$ such that, every completable G-partial matrix $a \in \mathbb{R}^{V \cup E}$ also has a psd completion of rank at most k. Notice that $\mathrm{gd}(\cdot)$ is well defined and upper bounded by the number of nodes of the graph, since any completable G-partial matrix has a completion of rank at most n.

In Chapter 5 we show that the graph parameter $\mathrm{gd}(\cdot)$ is minor monotone, i.e., if H is a minor of G then $\mathrm{gd}(H) \leq \mathrm{gd}(G)$. Our main result in Chapter 5 is to identify the list of forbidden minors for the graphs with small Gram dimension. Specifically,

- $\mathrm{gd}(G) \leq 2$ if and only if G has no K_3-minor,
- $\mathrm{gd}(G) \leq 3$ if and only if G has no K_4-minor,
- $\mathrm{gd}(G) \leq 4$ if and only if G has no K_5 and $K_{2,2,2}$-minors.

The main difficulty in this proof is to obtain the characterization of graphs having Gram dimension at most four and our approach consists of two main ingredients. The first one is to reduce the problem to the study of the two graphs V_8 and $C_5 \square K_2$ (cf. Figures 5.2 and 5.3) and to show that $\mathrm{gd}(V_8) \leq 4$ and $\mathrm{gd}(C_5 \square K_2) \leq 4$. To arrive at this result we rely on the forbidden minor characterization of graphs with treewidth at most 3 given in [11] and the fact that

$$\mathrm{gd}(G) \leq \mathrm{tw}(G) + 1, \tag{1.7}$$

for any graph G; cf. Lemma 5.2.8. The second ingredient is to construct partial matrices that admit a *unique* completion to a full psd matrix. This problem is explored in Chapter 11 where we obtain a sufficient condition for constructing such partial matrices. Furthermore, we establish interesting connections with the theory of universally rigid graphs.

Although the definition of the Gram dimension appears to be tailored to the psd matrix completion problem, it can also be used to bound the rank of optimal solutions to general SDP's. Namely, consider a semidefinite program (P) in canonical form (1.1) and recall that its aggregate sparsity pattern is the graph $\mathscr{A}_P = (V_P, E_P)$ where $V_P = [n]$ and whose edges correspond to positions where at least one of the matrices A_k $(k \in \{0, 1, \ldots, m\})$ has a nonzero entry. Then, we have the following easy but important fact:

1.2.1 Theorem. *Consider a semidefinite program in canonical primal form*

$$\begin{aligned} &\inf \langle A_0, X \rangle \\ &\textit{subject to } \langle A_k, X \rangle = b_k, \qquad k = 1, \ldots, m \\ &X \succeq 0. \end{aligned} \tag{P}$$

If (P) attains its optimum, it also has an optimal solution of rank at most $\mathrm{gd}(\mathscr{A}_P)$.

Proof. Let X^* be an optimal solution for (P) and consider the following completion problem:

$$X_{ij} = X^*_{ij} \text{ for all } ij \in E_P \text{ and } X \succeq 0. \tag{1.8}$$

Since (1.8) is feasible, it follows from the definition of the Gram dimension that it also has a feasible solution, say X', of rank at most $\mathrm{gd}(\mathscr{A}_P)$. By construction X' coincides with X^* for all $ij \in E_P$ which implies that X' is also optimal for (P). □

We remark that this fact remains true if one replaces some of the equations in (P) by inequalities. Indeed, by adding slack variables we get a new SDP in standard form, whose aggregate sparsity pattern is G with some additional isolated nodes, however with the same Gram dimension as G (cf. Lemma 5.2.7).

As an illustration, for a graph $G = ([n], E)$ consider the MAX CUT problem and its standard semidefinite programming relaxation given by

$$\max \frac{1}{4}\langle L_G, X\rangle \ \text{ s.t. } \ X_{ii} = 1 \ (i \in [n]), \ \ X \succeq 0, \tag{1.9}$$

where L_G denotes the Laplacian matrix of G.

Clearly, the aggregate sparsity pattern of the program (1.9) is equal to the graph G. Moreover, the optimal value of (1.9) is attained since the objective function is linear and the feasible region is compact. Consequently, Theorem 1.2.1 implies that the SDP (1.9) has an optimal solution of rank at most $\mathrm{gd}(G)$.

For SDP's of the form (1.9) we can improve on the $\mathrm{gd}(\cdot)$ bound. To achieve this, in Chapter 10 we introduce a graph parameter, called the extreme Gram dimension, which we denote by $\mathrm{egd}(\cdot)$. As we will see in Chapter 10, for any graph G, the SDP (1.9) has an optimal solution of rank at most $\mathrm{egd}(G)$. Furthermore, for any graph G we have that $\mathrm{egd}(G) \leq \mathrm{gd}(G)$, and in some cases this inequality can be strict. For example we have that $\mathrm{egd}(K_n) = \lfloor \frac{\sqrt{8n+1}-1}{2} \rfloor < n = \mathrm{gd}(K_n)$ for all $n \geq 2$ (cf. Lemma 10.1.5). Consequently, the bound given by $\mathrm{egd}(\cdot)$ may improve on the $\mathrm{gd}(\cdot)$ bound.

As a second example, given a graph $G = ([n], E)$ with weights $s \in \mathbb{R}^n_+$ and $w \in \mathbb{R}^E_+$, consider the following semidefinite programs

$$\min \sum_{i=1}^{n} s_i X_{ii} \ \text{ s.t. } \ X_{ii} + X_{jj} - 2X_{ij} \geq w_{ij} \ (ij \in E), \ X \succeq 0, \tag{1.10}$$

$$\max \sum_{i=1}^{n} s_i X_{ii} \ \text{ s.t. } \sum_{i,j=1}^{n} s_i s_j X_{ij} = 0, \ X_{ii} + X_{ij} - 2X_{ij} \leq w_{ij} \ (ij \in E), \ X \succeq 0. \tag{1.11}$$

These programs were studied respectively in [52] and [53] for their relevance to optimization problems concerning the eigenvalues of the weighted Laplacian of G. For both of these programs, it was shown in [52] and [53] respectively, that there exists an optimal solution of rank at most $\mathrm{tw}(G) + 1$. For program (1.10), this result also follows from our treewidth upper bound for the Gram dimension given in (1.7), since G is the aggregated sparsity pattern of (1.10). However, the aggregated sparsity pattern of program (1.11) could be much denser than G, so our treewidth bound on the Gram dimension does not apply to it.

In Chapter 6 we identify connections between the Gram dimension and two other graph parameters that have been studied in the literature. The first one is the Euclidean dimension of a graph $G = ([n], E)$ introduced already in Section 1.1.3. Recall that graphs with small Euclidean dimension were characterized in [25, 26]: $\mathrm{ed}(G) \leq 1$ if and only if G has no K_3-minor, $\mathrm{ed}(G) \leq 2$ if and only if G has no K_4-minor and $\mathrm{ed}(G) \leq 3$ if and only if has no K_5 and $K_{2,2,2}$-minors.

Comparing the characterization of graphs with small Euclidean dimension with the characterization of graphs with small Gram dimension suggests that these two graph parameters should be closely related. In Section 6.1 we show that

$$\mathrm{gd}(G) = \mathrm{ed}(\nabla G) \tag{1.12}$$

and that

$$\mathrm{ed}(\nabla G) \geq \mathrm{ed}(G) + 1, \tag{1.13}$$

where ∇G denotes the graph obtained by G by adding a new vertex which gets connected to all existing vertices.

By combining (1.12) and (1.13) we have that

$$\mathrm{ed}(G) \leq \mathrm{gd}(G) - 1,$$

for any graph G. This inequality, combined with the forbidden minor characterization for graphs with Gram dimension at most 4, implies that if G has no K_5 and $K_{2,2,2}$-minor then $\mathrm{ed}(G) \leq 3$. This implication is the difficult direction in the proof of [25, 26], so in this sense, our characterization of graphs with Gram dimension at most four implies the characterization of graphs with Euclidean dimension at most three. Determining whether (1.13) holds with equality remains an interesting open problem.

Furthermore, in Section 6.2 we establish interesting connections with the Colin de Verdière-type spectral graph parameter $\nu^{=}(\cdot)$, introduced in [126, 129]. This parameter is defined as the maximum corank of an n-by-n positive semidefinite matrix M such that $M_{ij} = 0$ for all $ij \notin E$ and moreover

$$X \in \mathscr{S}^n,\ MX = 0,\ X_{ij} = 0 \text{ for all } \{i,j\} \in V \cup E \Longrightarrow X = 0, \tag{1.14}$$

a property known as the *Strong Arnold Property* (cf. Section 3.4).

The study of the graph parameter $\nu^{=}(\cdot)$ is motivated by its relevance to the celebrated graph parameter $\mu(\cdot)$, introduced by Colin de Verdière [42]. In [129] it was shown that the parameter $\nu^{=}(\cdot)$ is minor monotone and moreover the graphs with small value of $\nu^{=}(\cdot)$ have been characterized:

- $\nu^{=}(G) \leq 2$ if and only if G has no K_3-minor.
- $\nu^{=}(G) \leq 3$ if and only if G has no K_4-minor.
- $\nu^{=}(G) \leq 4$ if and only if G has no K_5 and $K_{2,2,2}$-minors.

The fact that the characterizations for the graphs having small values for $\mathrm{gd}(\cdot)$ and $\nu^{=}(\cdot)$ coincide, suggests some relation between these two parameters. In Chapter 6 we show that for any graph G,

$$\mathrm{gd}(G) \geq \nu^{=}(G). \tag{1.15}$$

Notice that (1.15) combined with the forbidden minor characterizations for small values of the Gram dimension imply all the characterizations for the parameter $\nu^{=}(\cdot)$ mentioned above. Determining whether (1.15) holds with equality remains an open problem.

Another problem we take up in this thesis is to investigate the complexity aspects associated with the psd matrix completion problem. As we have already remarked, the complexity status of deciding the feasibility of (1.6) for a rational vector $a \in \mathbb{Q}^{V \cup E}$ is not fully understood. Our focus in this thesis is on the following decision problem: For a fixed integer $k \geq 1$, we are given as input a graph G and a rational vector $a \in \mathbb{Q}^{V \cup E}$ and the goal is to decide whether (1.6) has a feasible solution of rank at most k.

In Chapter 7 we address this question and show that this problem is NP-hard for any fixed $k \geq 2$ (the case $k = 1$ is easily seen to be polynomial time solvable). To prove hardness, we need to use different reductions for the cases $k \geq 3$ and $k = 2$. In the former case, the hardness result follows from known complexity results concerning the existence of orthonormal representations of graphs [100, 101]. In the latter case we exploit the relation between Gram and Euclidean realizations of graphs, as spelled out in (1.12). Additionally, we need to use well known hardness results concerning the embeddability of weighted graphs in Euclidean space [118].

1.2.2 Grothendieck-type semidefinite programs

Given a graph $G = ([n], E)$ and a vector $w = (w_{ij}) \in \mathbb{R}^E$ consider the following quadratic integer program over the hypercube:

$$\mathrm{ip}(G, w) = \max \sum_{ij \in E} w_{ij} x_i x_j \ \text{ s.t. } \ x_1, \dots, x_n \in \{\pm 1\}. \tag{1.16}$$

The study of this program is motivated, in particular, by the fact that it models the MAX CUT problem in ± 1 variables.

As the program (1.16) is NP-hard, it is important to obtain tractable relaxations for it. In this thesis, we focus on the canonical relaxation of (1.16) given by

$$\mathrm{sdp}(G, w) = \max \sum_{ij \in E} w_{ij} u_i^{\mathsf{T}} u_j \ \text{ s.t. } \ u_1, \dots, u_n \in \mathbb{S}^{n-1}, \tag{1.17}$$

where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$. Using the fact that a matrix is positive semidefinite if and only if it is the Gram matrix of some family of vectors, we have that (1.17) is a semidefinite program and it can be reformulated as follows:

$$\mathrm{sdp}(G, w) = \max_{ij \in E} \sum w_{ij} X_{ij} \ \text{ s.t. } \ X_{ii} = 1 \ (i \in [n]), \ X \succeq 0. \tag{1.18}$$

The quality of relaxation (1.17) is measured by its integrality gap, known as the *Grothendieck constant* of the graph G. Specifically, for a graph G, its Grothendieck constant, denoted by $\kappa(G)$, is defined as

$$\kappa(G) = \sup_{w \in \mathbb{R}^E} \frac{\mathrm{sdp}(G, w)}{\mathrm{ip}(G, w)}. \tag{1.19}$$

Alternatively, $\kappa(G)$ is equal to the smallest $K > 0$ such that:

$$\mathrm{sdp}(G, w) \leq K \cdot \mathrm{ip}(G, w), \ \text{for every } w \in \mathbb{R}^E.$$

The Grothendieck constant of a graph was introduced and studied in [8]. The special case where G is a complete bipartite graph was studied by A. Grothendieck, although in a quite different language [57].

The classical Grothendieck inequality states that $\sup_{n,m \in \mathbb{N}} \kappa(K_{n,m})$ is an absolute constant, known as *Grothendieck's constant* [57]. Despite significant efforts, computing the exact value of this constant has proven to be elusive [29]. Nevertheless, the classical Grothendieck inequality has numerous algorithmic implications such as cut-norm estimation [9], construction of Szemerédi partitions of graphs [9] and quantum information theory [32].

The dependence of the Grothendieck constant on the combinatorial structure of the graph is not fully understood. The most important result in this direction, shown in [8], is that

$$\Omega(\log \omega(G)) \leq \kappa(G) \leq O(\log \vartheta(\overline{G})). \tag{1.20}$$

Additionally, it was shown in [31] that

$$\kappa(G) \leq \frac{\pi}{2\log\left(\frac{1+\sqrt{(\vartheta(\overline{G})-1)^2+1}}{\vartheta(\overline{G})-1}\right)}.$$

The reader is referred to Chapter 8 for additional background concerning the Grothendieck constant of a graph and a more comprehensive introduction to the topic of Grothendieck inequalities.

As was already suggested in Section 1.1.2 (in the setting of the MAX CUT problem), identifying instances for which (1.18) admits a low-rank optimal solution can lead to improved approximation guarantees for the integer program (1.16). With this in mind, for any fixed integer $r \geq 1$, we consider the rank-constrained SDP:

$$\mathrm{sdp}_r(G,w) = \max_{ij\in E} \sum w_{ij} X_{ij} \ \text{ s.t. } \ X_{ii} = 1 \ (i \in [n]), \ \mathrm{rank} X \leq r, \ X \succeq 0, \tag{1.21}$$

or equivalently,

$$\mathrm{sdp}_r(G,w) = \max \sum_{ij\in E} w_{ij} u_i^{\mathsf{T}} u_j \ \text{ s.t. } \ u_1,\ldots,u_n \in \mathbb{S}^{r-1}. \tag{1.22}$$

The study of programs of the form (1.22) is of significant interest in its own right, the main motivation coming from statistical mechanics and in particular from the *r-vector model* introduced by Stanley [124]. This model consists of an interaction graph $G = (V,E)$, where vertices correspond to particles and edges indicate whether there is interaction (ferromagnetic or antiferromagnetic) between the corresponding pair of particles. Additionally, there is a potential function $A : V \times V \mapsto \mathbb{R}$ satisfying $A_{ij} = 0$ if $ij \notin E$, $A_{ij} > 0$ if there is ferromagnetic interaction between i and j and $A_{ij} < 0$ if there is antiferromagnetic interaction between i and j. Additionally, particles possess a vector valued *spin* given by a function $f : V \mapsto \mathbb{S}^{r-1}$. Assuming that there is no external field acting on the system, its total energy is given by the *Hamiltonian* defined as

$$H(f) = -\sum_{ij\in E} A_{ij} f(i)^{\mathsf{T}} f(j).$$

A *ground state* is a configuration of spins that minimizes the Hamiltonian. The case $r = 1$ corresponds to the *Ising model*, the case $r = 2$ corresponds to the *XY model* and the case $r = 3$ to the *Heisenberg model*.

Consequently, calculating the Hamiltonian and computing ground states in any of these models amounts to solving a rank-constrained semidefinite program of the form (1.22). As the rank function is non-convex and non-differentiable, such problems are in general computationally challenging and this motivates the need for identifying tractable instances for (1.22).

Our contributions

As already mentioned, in Chapter 7 we address the complexity of the low-rank psd matrix completion problem. The second problem we consider in Chapter 7 is to investigate the complexity aspects of the rank-constrained semidefinite program (1.21). For this, it is useful to rewrite program (1.21) as

$$\mathrm{sdp}_r(G,w) = \max \sum_{ij\in E} w_{ij}X_{ij} \ \text{ s.t. } \ X \in \mathscr{E}_{n,r}, \tag{1.23}$$

where we define

$$\mathscr{E}_{n,r} = \{X \succeq 0 : \mathrm{rank} X \leq r,\ X_{ii} = 1\ (i \in [n])\},$$

known as the rank-constrained elliptope. Since the objective function in (1.23) is linear, program (1.23) corresponds to optimization over the convex hull of $\mathscr{E}_{n,r}$, i.e.,

$$\mathrm{sdp}_r(G,w) = \max \sum_{ij\in E} w_{ij}X_{ij} \ \text{ s.t. } \ X \in \mathrm{conv}(\mathscr{E}_{n,r}). \tag{1.24}$$

Let $\pi_E : \mathscr{S}^n \mapsto \mathbb{R}^E$ denote the projection from the set of n-by-n symmetric matrices $\mathscr{S}^n$ onto the subspace indexed by the edge set of G, i.e., $\pi_E(X) = (X_{ij})_{ij\in E}$. Since the objective function in (1.24) is a weighted linear combination of entries that correspond to edges of G, program (1.24) can be reformulated as follows:

$$\mathrm{sdp}_r(G,w) = \max \sum_{ij\in E} w_{ij}x_{ij} \ \text{ s.t. } \ x \in \pi_E(\mathrm{conv}(\mathscr{E}_{n,r})). \tag{1.25}$$

In the case $r = 1$, the feasible region of (1.25) is equal to the cut polytope of the graph G in ± 1 variables (cf. Chapter 4). This implies that in the case $r = 1$ program (1.25) is NP-hard. It is believed that (1.25) is also NP-hard for any fixed integer $r \geq 2$ (cf., e.g., the quote of Lovász [90, p. 61]).

For any $r \geq 2$, the feasible region of (1.25) is in general non-polyhedral. Hence, the right question to ask is about the complexity of the *weak* optimization problem. It follows from general results about the ellipsoid method (see, e.g., [58] for details) that the weak optimization problem and the weak membership problem in the feasible region of (1.25) have the same complexity status. In Chapter 7 we show that the strong membership problem in $\pi_E(\mathrm{conv}(\mathscr{E}_{n,r}))$ is NP-hard, thus providing some evidence of hardness of optimization of (1.25).

In Chapter 9 we consider the problem of calculating the Grothendieck constant of some specific graph classes. Our main result is a closed-form formula for the Grothendieck constant of graphs with no K_5-minor. Specifically, we show that if G has no K_5-minor and is not a forest then

$$\kappa(G) = \frac{g}{g-2}\cos\frac{\pi}{g},$$

where g denotes the girth of the graph G, i.e., the length of the shortest cycle contained in the graph. This result relies on the existence of explicit descriptions for the cut polytope and the elliptope of circuits and graphs with no K_5-minor and the fact that $\kappa(C_n) = \frac{n}{n-2}\cos(\pi/n)$, where C_n denotes the circuit graph on n nodes.

For the complete graph K_n, it follows from (1.20) that $\kappa(K_n) = \Theta(\log n)$. In view of (1.19) it is an interesting question to identify explicit inequalities that achieve this integrality gap. This question was posed as an open problem in [8] and some inequalities with large integrality gap have been identified in [13]. In Chapter 9 we show that for the class of clique-web inequalities, a wide class of valid inequalities for the cut polytope, the integrality gap is constant.

In Chapter 10 we consider the problem of identifying guarantees for the existence of low-rank optimal solutions to the semidefinite program (1.18). By definition of the parameter $\mathrm{gd}(\cdot)$ it follows that, for any $w \in \mathbb{R}^E$, program (1.18) has an optimal solution of rank at most $\mathrm{gd}(G)$. Our main goal in Chapter 10 is to show that for SDP's of the form (1.18), the $\mathrm{gd}(G)$ bound can be improved.

To achieve this, for any integer $r \geq 1$, we consider the graphs with the property that (1.18) has an optimal solution of rank at most r, for all $w \in \mathbb{R}^E$. Equivalently, for any fixed $r \geq 1$, we consider the graphs with the following property:

$$\mathrm{sdp}_r(G,w) = \mathrm{sdp}(G,w) \quad \text{for all} \quad w \in \mathbb{R}^E, \tag{1.26}$$

where $\mathrm{sdp}_r(G,w)$ was defined in (1.21). Then, in Chapter 10 we introduce a new graph parameter called *the extreme Gram dimension of a graph*, which we denote by $\mathrm{egd}(\cdot)$. The extreme Gram dimension of a graph G is defined as the smallest integer $r \geq 1$ for which (1.26) holds. Notice that (1.26) is valid for $r = |V(G)|$ and thus the extreme Gram dimension is well defined and upper bounded by the number of nodes of the graph.

As suggested by the names of the two parameters, the Gram dimension and the extreme Gram dimension of a graph are closely related. Our first goal in Chapter 10 is to describe the precise nature of this relationship. In particular, we determine a reformulation for the extreme Gram dimension which shows that for any graph G,

$$\mathrm{egd}(G) \leq \mathrm{gd}(G).$$

Moreover, in some cases this inequality can be strict, e.g., for the complete graph K_n (cf. Lemma 10.1.5).

In Chapter 10 we show that the parameter $\mathrm{egd}(\cdot)$ is minor-monotone. Consequently, for any fixed integer $r \geq 1$, the graphs satisfying $\mathrm{egd}(G) \leq r$ can be characterized by a finite list of minimal forbidden minors. For the case $r = 1$ it is known that the only forbidden minor is the graph K_3 [74]. Our main result in Chapter 10 is to identify the list of minimal forbidden minors for the case $r = 2$.

Moreover, in Chapter 10 we introduce a new treewidth-like graph parameter, denoted by $\mathrm{la}_{\boxtimes}(\cdot)$, which we call the *strong largeur d'arborescence*. The parameter $\mathrm{la}_{\boxtimes}(G)$ is defined as the smallest integer $r \geq 1$ such that G is a minor of the strong graph product $T \boxtimes K_r$, where T is a tree and K_r denotes the complete graph on r vertices. The name of the parameter is derived from the related parameter *largeur d'arborescence* introduced by Colin de Verdière, where the strong product is replaced by the Cartesian product of graphs [43]. In Chapter 10 we investigate the properties of this parameter and show that the extreme Gram dimension of a graph is upper bounded by its strong largeur d'arborescence.

1.2.3 Partial matrices with a unique psd completion

A crucial ingredient in the study of the Gram dimension and the extreme Gram dimension, is the ability to construct partial positive semidefinite matrices that admit a unique completion to a full positive semidefinite matrix. In Chapters 5 and 10 we present several such constructions, but the proofs there are mainly by direct case checking. In Chapter 11 we obtain a sufficient condition which provides us with a systematic way for constructing such partial matrices. Using this condition, we can recover most examples from Chapters 5 and 10.

The condition for uniqueness of a psd completion suggests a connection to the theory of universal rigidity. A *framework* consists of a graph $G = ([n], E)$ and an assignment of vectors $\mathbf{p} = \{p_1, \ldots, p_n\}$ to the nodes of the graph, and is denoted by $G(\mathbf{p})$. A framework $G(\mathbf{p})$ is said to be *universally rigid* if it is the only framework having the same edge lengths in any space, up to congruence. A related concept is that of global rigidity of frameworks. A framework $G(\mathbf{p})$ in $\mathbb{R}^d$ is called *globally rigid in* $\mathbb{R}^d$ if, up to congruence, it is the only framework in $\mathbb{R}^d$ having the same edge lengths. These concepts have been extensively studied and there exists rich literature about them (see e.g. [33, 34, 35, 36, 54] and references therein).

The analogue of the notion of global rigidity, in the case when Euclidean distances are replaced by inner products, was recently investigated in [122]. There, it is shown that many of the results that are valid in the setting of Euclidean distances can be adapted to the so-called 'spherical setting'. The latter terminology refers to the fact that, when the vectors $p_1, \ldots, p_n \in \mathbb{R}^d$ are restricted to lie on the unit sphere, their pairwise inner products lead to the study of the spherical metric space, where the distance between two points p_i, p_j is given by $\arccos(p_i^\mathsf{T} p_j)$, i.e., the angle formed between the two vectors [117].

Taking this analogy further, our sufficient condition for constructing partial positive semidefinite matrices with a unique psd completion can be interpreted as the spherical analogue of Connelly's celebrated sufficient condition for the universal rigidity of frameworks. In Chapter 11 we compare these two sufficient conditions and show that they are equivalent for a special class of frameworks.

Publications and Preprints

The content of this thesis is based on the following publications and preprints:

- M. Laurent and A. Varvitsiotis. Computing the Grothendieck constant of some graph classes. Operations Research Letters, 39(6):452-456, 2011.
- M. Laurent and A. Varvitsiotis. A new graph parameter related to bounded rank positive semidefinite matrix completions. *Mathematical Programming Series A*, Published online: February 2013.
- M. E.-Nagy, M. Laurent, and A. Varvitsiotis. Complexity of the positive semidefinite matrix completion problem with a rank constraint. In K. Bezdek, A. Deza, and Y. Ye, editors, *Discrete Geometry and Optimization*, vol. 69 of Fields Institute Communications, pages 105–120, Springer, 2013.
- M. E.-Nagy, M. Laurent, and A. Varvitsiotis. On bounded rank positive semidefinite matrix completions of extreme partial correlation matrices, 2012. Accepted for publication with minor revisions in Journal of Combinatorial Theory, Series B. Available at: arXiv:1205.2040.
- M. Laurent and A. Varvitsiotis. Positive semidefinite matrix completion, universal rigidity and the Strong Arnold Property, January 2013. Available at: arXiv:1301.6616.

2 Preliminaries

In this section we introduce some basic notions from convexity, linear algebra and graph theory that are relevant for this thesis. With the intention to maximize the readability of the thesis we will reintroduce these definitions whenever necessary.

2.1 Convexity

A set $C \subseteq \mathbb{R}^n$ is called *convex* if $\lambda c + (1-\lambda)c' \in C$ for every $c, c' \in C$ and $\lambda \in [0,1]$. A *hyperplane* is an affine subspace of $\mathbb{R}^n$ of codimension 1 and has the form

$$\{x \in \mathbb{R}^n : c^\mathsf{T} x = b\},$$

where $c \in \mathbb{R}^n, c \neq 0$ and $b \in \mathbb{R}$. A hyperplane H partitions the space into two closed *halfspaces*

$$H^+ = \{x \in \mathbb{R}^n : c^\mathsf{T} x \geq b\} \text{ and } H^- = \{x \in \mathbb{R}^n : c^\mathsf{T} x \leq b\}.$$

We say that the hyperplane H *supports* the set C at the point $x \in C$ if $x \in H$ and C is contained in one of the halfspaces H^+ or H^-. A hyperplane is said to *separate* two convex sets C and C' if C lies in one of the closed halfspaces determined by H and C' lies in the other. Moreover, H is said to separate C and C' *properly* if it separates them, but not both of them lie in H.

A set $C \subseteq \mathbb{R}^n$ is called a *cone* if $0 \in C$ and $\lambda x \in C$ for every scalar $\lambda \geq 0$ and every $x \in C$. Given a subset $C \subseteq \mathbb{R}^n$, its *polar* is the set

$$C^\circ = \{x \in \mathbb{R}^n : x^\mathsf{T} y \leq 1, \text{ for all } y \in C\}.$$

2.1.1 Lemma. *For a polyhedron $P = \{x \in \mathbb{R}^n : a_i^\mathsf{T} x \leq 1 \ (i \in [m])\}$ we have that*

$$P^\circ = \operatorname{conv}(0, a_1, \ldots, a_m).$$

Given a subset $C \subseteq \mathbb{R}^n$, its *dual cone* is the set

$$C^* = \{x \in \mathbb{R}^n : x^\top y \geq 0, \text{ for all } y \in C\}.$$

Notice that C^* is always a convex cone even if that is not the case for C. A cone is called *self-dual* if $C = C^*$.

The following well known lemma from convex analysis identifies appropriate conditions to achieve proper separation between two convex sets.

2.1.2 Lemma. *Every pair of non-empty convex sets in $\mathbb{R}^n$ whose relative interiors are disjoint can be properly separated by a hyperplane in $\mathbb{R}^n$.*

The following refinement of Lemma 2.1.2 will also be useful.

2.1.3 Lemma. *Consider a pair of non-empty convex subsets of $\mathbb{R}^n$ and assume that they can be properly separated by a hyperplane. If (at least) one of them is a cone then there exists a hyperplane which properly separates them and passes through the origin.*

A convex subset $F \subseteq C$ is called a *face* of C if, for any $x, y \in C$, $\lambda x + (1-\lambda)y \in F$ for some scalar $\lambda \in (0,1)$ implies $x, y \in F$. The *exposed faces* of a convex set C are the sets of the form $C \cap H$, where H is a supporting hyperplane to C. For an element $x \in C$ we denote by $F_C(x)$ the smallest face of C containing x. As the intersection of two faces of C is also a face of C it follows that $F_C(x)$ is well defined.

2.1.4 Lemma. *Let C be a convex subset of $\mathbb{R}^n$ and let $x \in C$. Then $F_C(x)$ is the unique face of C that contains x in its relative interior.*

Proof. Assume for contradiction that $x \notin \operatorname{relint} F_C(x)$. Then, Lemma 2.1.2 implies that the two convex sets $\{x\}$ and $F_C(x)$ can be properly separated by a hyperplane H, i.e., here exists a nonzero vector $c \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$ such that $c^\top x \geq b$ and $c^\top y \leq b$ for all $y \in F_C(x)$. Since $x \in F_C(x)$ it follows that $c^\top x = b$ and thus $x \in H$ and since the separation is proper we get that $F_C(x) \setminus H \neq \emptyset$. We arrive at a contradiction by noticing that $F_C(x) \cap H$ is a face of C containing x and is strictly contained in $F_C(x)$. Lastly, we show that $F_C(x)$ is the unique face of C containing x in its relative interior. For this let F be a face of C with $x \in \operatorname{relint} F$. Clearly, we have that $F_C(x) \subseteq F$. For the other inclusion let $y \in F$. As $x \in \operatorname{relint} F$ there exists a point $z \in F$ and a scalar $\lambda \in (0,1)$ such that $x = \lambda y + (1-\lambda)z$. Since $x \in F_C(x)$ and $F_C(x)$ is a face of C we get that $y \in F_C(x)$. $\square$

A point $x \in C$ is called an *extreme point* of C if and only if $F_C(x) = \{x\}$. The set of extreme point of a convex set C is denoted by $\operatorname{ext} C$.

2.1.5 Lemma. *Let C be a convex set and let F be a face of C. Then* $\operatorname{ext} F \subseteq \operatorname{ext} C$.

A vector $z \in V$ is said to be a *perturbation* of $x \in C$ if $x \pm \epsilon z \in C$ for some $\epsilon > 0$. The set of perturbations of $x \in C$ form a linear space which we denote as $\operatorname{Pert}_C(x)$ and the dimension of $F_C(x)$ is equal to the dimension of $\operatorname{Pert}_C(x)$ as a linear space.

2.2 Graph theory

Graphs are structures which are ubiquitous in computer science and mathematics and are useful for modeling various networks like the internet and social networks. In the next two sections we introduce all necessary definitions and concepts that are relevant for this thesis.

2.2.1 Basic definitions

Throughout this thesis, $[n]$ denotes the set $\{1,\ldots,n\}$. Given a graph $G=(V,E)$, we also denote its node set by $V(G)$ and its edge set by $E(G)$. A *component* is a maximal connected subgraph of G. A *cutset* is a subset of nodes $U \subseteq V$ with the property that the graph obtained by deleting the nodes in U has more connected components than G. A cutset of is called a *cut node* if $|U|=1$, and G is *2-connected* if it is connected and has no cut node. For $U \subseteq V$, $G[U]$ is the subgraph induced by U. Given $\{u,v\} \notin E(G)$, we denote by $G+\{u,v\}$ the graph obtained by adding the edge $\{u,v\}$ to G.

A *clique* in G is a set of pairwise adjacent nodes and $\omega(G)$ denotes the maximum cardinality of a clique in G. A *k-clique* is a clique of cardinality k. Let $G=(V,E)$, $G'=(V',E')$ be two graphs, where $V \cap V'$ is a clique in both G and G'. Their *clique sum* is the graph $G=(V \cup V', E \cup E')$, also called their *clique k-sum* when $k=|V_1 \cap V_2|$.

We denote the complete graph on n nodes by K_n. A *cycle* in a graph is a sequence of vertices starting and ending at the same vertex, where each two consecutive vertices in the sequence are adjacent to each other. A *circuit* in a graph is a cycle where no repetitions of vertices or edges is allowed, other than the repetition of the starting and ending vertex. Throughout this thesis we denote by C_n the circuit on n nodes. If C is a circuit in G, a *chord* of C is an edge $\{u,v\} \in E$ where u and v are two nodes of C that are not consecutive on C. A graph G is said to be *chordal* if every circuit of length at least 4 has a chord. As is well known, a graph G is chordal if and only if G is a clique sum of cliques.

For a graph G, we denote by ∇G its *suspension graph*, obtained by adding a new node, called the *apex* node, which is adjacent to all nodes of G. Moreover, we denote by $\nabla^p G$ the graph by iteratively applying the suspension operation p-times.

Given an edge $e=\{u,v\} \in E$, $G \backslash e = (V, E \setminus \{e\})$ is the graph obtained from G by *deleting* the edge e and G/e is obtained by *contracting* the edge e: Replace the two nodes u and v by a new node, adjacent to all the neighbors of u and v. A graph M is a *minor* of G, denoted as $M \preceq G$, if M can be obtained from G by a series of edge deletions and contractions and isolated node deletions, ignoring any loops or multiple edges that may arise. Equivalently, M is a minor of a connected graph G if there is a partition of $V(G)$ into nonempty subsets $\{V_i : i \in V(M)\}$ where each $G[V_i]$ is connected and, for each edge $\{i,j\} \in E(M)$, there exists at least one edge in G between V_i and V_j. Then the collection $\{V_i : i \in V(M)\}$ is called an *M-partition* of G and the V_i's are its *classes*.

Given a finite list $\mathscr{M}$ of graphs, $\mathscr{F}(\mathscr{M})$ denotes the collection of all graphs that do not admit any graph in $\mathscr{M}$ as a minor. By the celebrated graph minor theorem of Robertson and Seymour [115], any family of graphs which is closed under the operation of taking minors is of the form $\mathscr{F}(\mathscr{M})$ for some finite set $\mathscr{M}$ of graphs. In this setting, closed means that every minor of a graph in the family is also contained in the family. The set $\mathscr{M}$ is called the *obstruction set* of the class.

The archetypical example for this, that also served as the main motivation for the graph minor theorem, is the class of planar graphs. Planar graphs are closed under the operation of taking minors, so the graph minor theorem asserts the existence of a finite list of forbidden minors. The obstruction set was determined already in 1930 by Kuratowski, and this characterization is today known as Kuratowski's theorem: A graph G is planar if and only if G does not have a K_5 or a $K_{3,3}$-minor [73].

Deciding whether a graph H is a minor of a graph G is an NP-complete problem when both graphs are part of the input. Indeed, consider the case when H is a circuit with the same number of nodes as G. Then, deciding whether $H \preceq G$ is equivalent to deciding whether G contains a Hamilton cycle. On the other hand if the graph H is fixed, this can be decided in time $O(|V(H)|^3)$ [114]. This implies that for any minor closed class of graphs there *exists* a polynomial time algorithm for deciding membership in the class. Nevertheless, to use this algorithm, we first need to determine the obstruction set, a task which is usually very challenging.

A *graph parameter* is any function from the set of graphs (up to isomorphism) to the complex numbers. In this thesis we restrict to graph parameters that take values in the natural numbers. A graph parameter $f(\cdot)$ is called *minor monotone* if

$$f(G\backslash e) \leq f(G) \text{ and } f(G/e) \leq f(G),$$

for any graph G and any edge e of G.

Notice that given a minor monotone graph parameter $f(\cdot)$ and a fixed integer $k \geq 1$, the family of graphs G satisfying $f(G) \leq k$ is closed under taking minors. By the preceding discussion it follows that for any fixed integer $k \geq 1$ there exists a forbidden minor characterization for the family of graphs satisfying $f(G) \leq k$.

A *homeomorph* (or subdivision) of a graph M is obtained by replacing its edges by paths. When M has maximum degree at most 3, G admits M as a minor if and only if it contains a homeomorph of M as a subgraph.

The *Cartesian product* of two graphs $G = (V, E)$ and $G' = (V', E')$, denoted by $G \square G'$, is the graph with node set $V \times V'$ and distinct nodes $(i, i'), (j, j') \in V \times V'$ are adjacent in $G \square G'$ when $i = j$ and $(i', j') \in G'$, or $(i, j) \in G$ and $i' = j'$.

The *strong product* of two graphs $G = (V, E)$ and $G' = (V', E')$, denoted by $G \boxtimes G'$, is the graph with node set $V \times V'$ and distinct nodes $(i, i'), (j, j') \in V \times V'$ are adjacent in $G \boxtimes G'$ when $i = j$ or $(i, j) \in E$, and $i' = j'$ or $(i', j') \in E'$.

The *tensor product* of two graphs $G = (V, E)$ and $G' = (V', E')$, denoted by $G \times G'$ is the graph with node set $V \times V'$ and distinct nodes $(i, i'), (j, j') \in V \times V'$ are adjacent in $G \times G'$ when $(i, j) \in E$ and $(i', j') \in E'$.

2.2.2 Width parameters

Several combinatorial optimization problems that are NP-hard for general graphs can be solved efficiently when the input graph is restricted to be a tree. It is reasonable to expect similar behavior for graphs that resemble trees.

In this section we introduce two graph parameters that quantify the resemblance of a graph with a tree. The first such parameter was introduced by Robertson and Seymour in their fundamental work on graph minors [115] and is commonly used in the parameterized complexity analysis of graph algorithms.

2.2.1 Definition. *The* treewidth *of a graph G, denoted by* $\mathrm{tw}(G)$*, is the smallest integer k such that G is contained in a clique k-sum of copies of K_{k+1}.*

The graphs with treewidth at most k are also called *partial k-trees*. Calculating the treewidth of a graph is NP-hard [10]. On the other hand, for any fixed integer $k \geq 1$, deciding whether the treewidth of a graph is at most k can be done in linear-time [28].

It is easy to see that a minor of a partial k-tree is also a partial k-tree. Consequently, for any fixed value of $k \geq 1$, the graphs with treewidth at most k can be characterized by a finite list of forbidden minors. Specifically, it is known that

2.2.2 Theorem. *[11, 45] For any graph G we have that*

(i) $\mathrm{tw}(G) = 1$ *if and only if G has no* K_3*-minor.*

(ii) $\mathrm{tw}(G) \leq 2$ *if and only if G has no* K_4*-minor.*

(iii) $\mathrm{tw}(G) \leq 3$ *if and only if G has no* $K_5, K_{2,2,2}, V_8$ *or* $C_5 \square K_2$*-minor.*

The graph V_8 is known as the *Wagner graph* or the Möbius ladder on 8 nodes; cf. Figure 5.2. The graph $K_{2,2,2}$ is the complete tripartite graph where all three partitions have cardinality 2.

Another useful property of the treewidth is that it behaves nicely with respect to the clique-sum operation.

2.2.3 Lemma. *If G is obtained as the clique sum of* G_1 *and* G_2 *then*

$$\mathrm{tw}(G) = \max\{\mathrm{tw}(G_1), \mathrm{tw}(G_2)\}.$$

We continue with a second width parameter that was introduced by Colin de Verdière in relation to the celebrated $\mu(\cdot)$ graph invariant [43].

2.2.4 Definition. *The* largeur d' arborescence *of a graph G, denoted by* $\mathrm{la}_{\square}(G)$*, is the smallest integer* $k \geq 1$ *for which G is a minor of* $T \square K_k$ *for some tree T.*

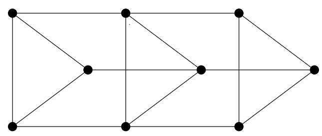

Figure 2.1: The graph $P_3 \square K_3$.

It turns out that the parameter $\mathrm{la}_{\square}(\cdot)$ is closely related to the notion of treewidth as illustrated in the following theorem.

2.2.5 Theorem. *[43, 126] For any graph G we have that*

$$\mathrm{tw}(G) \leq \mathrm{la}_{\square}(G) \leq \mathrm{tw}(G) + 1.$$

Proof. The lower bound follows easily since the graph $T \square K_k$ can be seen as the clique k-sum of copies of $K_2 \square K_k$. This observation combined with Lemma 2.2.3 and the fact that $\mathrm{tw}(K_2 \square K_k) = k$ implies the claim. The proof of the upper bound is more involved and the reader is referred to [43] for a detailed proof. □

Recall that it is NP-hard to obtain an *additive* ρ-approximation for the treewidth of a graph for any fixed constant ρ [68, Theorem 6.3.1]. Then, Theorem 2.2.5 implies that determining the value of the parameter $\mathrm{la}_{\square}(\cdot)$ is NP-hard.

The following easy lemma allows us to restrict the study of the parameter $\mathrm{la}_{\square}(\cdot)$ to 2-connected graphs.

2.2.6 Lemma. *[70] If G is the disjoint union or the 1-sum of graphs* G_1 *and* G_2 *then*

$$\mathrm{la}_{\square}(G) = \max\{\mathrm{la}_{\square}(G_1), \mathrm{la}_{\square}(G_2)\}.$$

It is clear from the definition that the parameter $\mathrm{la}_{\square}(\cdot)$ is minor monotone. The list of minimal forbidden minors has been determined for the first small values of the parameter as illustrated in the following theorem.

2.2.7 Theorem. *For a 2-connected graph G we have that:*

(i) $\mathrm{la}_{\square}(G) = 1$ *if and only if* G *has no* K_3*-minor.*

(ii) $\mathrm{la}_{\square}(G) \leq 2$ *if and only if* G *has no* K_4 *or* F_3*-minor.*

The graph F_3 is illustrated in Figure 10.1. The first part of Theorem 2.2.7 was shown in [43] and the second one in [70] and [126]. The case $k = 3$ is more involved and for the full list of forbidden minors the reader is referred to [128].

We note in passing that in Chapter 10 we study a variant of the strong largeur d'arborescence where the Cartesian product is replaced with the strong graph product; cf. Definition 10.1.8. This parameter will play an important role in Chapter 10.

2.3 The cone of positive semidefinite matrices

2.3.1 Basic definitions

Throughout this thesis we denote by $\mathcal{S}^n$ the set of n-by-n symmetric matrices. A matrix $A \in \mathcal{S}^n$ is called *positive semidefinite* (psd) if the associated quadratic form $x^{\mathsf{T}}Ax$ is nonnegative, i.e., $x^{\mathsf{T}}Ax \geq 0$ for every $x \in \mathbb{R}^n \setminus \{0\}$, and we denote by $\mathcal{S}^n_+$ the set of n-by-n positive semidefinite matrices. Whenever convenient we will also use the notation $X \succeq 0$.

For a matrix $A \in \mathcal{S}^n$ we denote its column space by $\mathrm{Range}A$ and its dimension, known as the rank of the matrix, is denoted by $\mathrm{rank}A$. Furthermore, its kernel is denoted by $\mathrm{Ker}A$ and the dimension of the kernel, known as the *corank* of the matrix, is denoted by $\mathrm{corank}A$.

The *Gram matrix* of a set of vectors $p_1, \ldots, p_n \in \mathbb{R}^k$, denoted by $\mathrm{Gram}(p_1, \ldots, p_n)$ is the n-by-n matrix whose (i, j) entry is given by $p_i^{\mathsf{T}} p_j$. For every vector $x \in \mathbb{R}^n$ we have that $x^{\mathsf{T}}\mathrm{Gram}(p_1, \ldots, p_n)x = \| \sum_{i=1}^n x_i p_i \|^2$ which implies that the Gram matrix of any family of vectors is positive semidefinite. Furthermore, if we arrange the vectors $p_1, \ldots, p_n$ as the columns of a k-by-n matrix R we see that $\mathrm{Gram}(p_1, \ldots, p_n) = R^{\mathsf{T}}R$ and thus $\mathrm{rank}\,\mathrm{Gram}(p_1, \ldots, p_n) = \mathrm{rank}R^{\mathsf{T}}R = \mathrm{rank}R = \dim\langle p_1, \ldots, p_n\rangle$.

There are several equivalent reformulations for a matrix to be positive semidefinite and the most important ones are summarized in the following theorem.

2.3.1 Theorem. *Consider a matrix $A \in \mathcal{S}^n$. The following statements are equivalent:*

(i) A is positive semidefinite.

(ii) All its eigenvalues are nonnegative.

(iii) All $2^n - 1$ *principal minors are positive semidefinite.*

(iv) There exists a matrix R such that $A = R^{\mathsf{T}}R$.

(v) A is the Gram matrix of some family of vectors.

A matrix $A \in \mathcal{S}^n$ is called *positive definite* if the associated quadratic form $x^{\mathsf{T}}Ax$ is strictly positive, i.e., $x^{\mathsf{T}}Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, and we denote by $\mathcal{S}^n_{++}$ the set of n-by-n positive definite matrices. Additionally, we also use the notation $A \succ 0$.

2.3.2 Theorem. *Consider a matrix $A \in \mathcal{S}^n$. The following statements are equivalent:*

(i) A is positive definite.

(ii) All its eigenvalues are strictly positive.

(iii) All the leading principal minors are positive definite.

(iv) There exists a matrix R with n independent columns such that $A = R^\top R$.

(iv) A is the Gram matrix of a family of n linearly independent vectors.

Throughout this thesis we assume that the linear space $\mathcal{S}^n$ is equipped with the trace inner product given by $\langle X, Y\rangle = \text{trace}(XY) = \sum_{i,j=1}^n X_{ij}Y_{ij}$. In turn, the trace inner product induces a norm on the space $\mathcal{S}^n$, known as the *Frobenious norm*, which is defined as $\|A\|_F = \sqrt{\text{trace}(A^2)}$.

From the real spectral theorem, for any matrix $A \in \mathcal{S}^n$, there exists an orthonormal basis of eigenvectors, i.e, there exists an orthogonal matrix $Q \in O(n)$ such that $A = Q\Lambda Q^\top$, where Λ is a diagonal matrix whose entries are eigenvalues of A.

2.3.2 Geometry of the psd cone

The geometry of the cone of positive semidefinite matrices has been studied extensively and is very well understood. In the following theorem we summarize some properties that are relevant for this thesis.

2.3.3 Theorem. *The following properties hold for the cone of positive semidefinite matrices:*

(i) $\mathcal{S}^n_+$ is a closed convex cone.

(ii) $\text{int}\, \mathcal{S}^n_+ = \mathcal{S}^n_{++}$.

(iii) $\mathcal{S}^n_+$ is a self-dual cone, i.e.,

$$X \in \mathcal{S}^n_+ \text{ if and only if } \langle X, Y\rangle \geq 0, \text{ for all } Y \in \mathcal{S}^n_+. \tag{2.1}$$

(iv) The extreme rays of $\mathcal{S}^n_+$ are generated by the rank one matrices, i.e., every extreme ray is of the form $\{\lambda x x^\top : \lambda \geq 0\}$ for some $x \in \mathbb{R}^n$.

Quoting A. Barvinok, "The cone of positive semidefinite matrices is arguably the most important of all non-polyhedral cones whose facial structure we completely understand." In the following theorem we recall the most important facts concerning the facial structure of the cone of psd matrices.

2.3.4 Theorem. *Consider a matrix $A \in \mathcal{S}^n_+$ and assume that* $\text{rank} A = r$. *Then, the smallest face of $\mathcal{S}^n_+$ containing A is given by*

$$F_{\mathcal{S}^n_+}(A) = \{X \in \mathcal{S}^n_+ : \text{Ker} A \subseteq \text{Ker} X\}.$$

Moreover, the face $F_{\mathcal{S}^n_+}(A)$ is linearly isomorphic to $\mathcal{S}^r_+$.

Proof. Let $u_1,\dots,u_r$ be an orthonormal basis for the range of A and we extend it to an orthonormal basis for $\mathbb{R}^n$, say $u_1,\dots,u_r,u_{r+1},\dots,u_n$. Let $U\in\mathbb{R}^{n\times n}$ be the orthogonal matrix whose columns are given by this orthonormal basis. Then, $A=UDU^{\mathsf{T}}=UDU^{-1}$ where $D=\operatorname{diag}(\lambda_1,\dots,\lambda_r,0,\dots,0)$. Let $C=\operatorname{diag}(0,\dots,0,1,\dots,1)$ where the first r entries are equal to zero and the rest $n-r$ entries are equal to one and set $Q=UCU^{-1}$. Clearly, $Q\in\mathcal{S}^n_+$ and $\langle Q,A\rangle=0$ and thus the hyperplane $H=\{X\in\mathcal{S}^n_+:\langle Q,X\rangle=0\}$ supports the cone $\mathcal{S}^n_+$ at A. This implies that the set $F=\mathcal{S}^n_+\cap H$ is an face of $\mathcal{S}^n_+$. For a matrix $X\in F$ we have that

$$\langle Q,X\rangle=0\Longleftrightarrow\sum_{i=r+1}^n u_i^{\mathsf{T}}Xu_i=0\Longleftrightarrow Xu_i=0,$$

for all $r+1\le i\le n$. This shows that $F=\{X\in\mathcal{S}^n_+:\operatorname{Ker}A\subseteq\operatorname{Ker}X\}$.

Our next goal is to show that F is linearly isomorphic to $\mathcal{S}^r_+$. For this consider the map $T:\mathcal{S}^n_+\mapsto\mathcal{S}^n_+$ defined as $T(X)=U^{-1}XU$. The map T is a linear bijection that maps the face F to the face $T(F)=\{U^{-1}XU:X\in F\}$. One can easily verify that $T(F)=\{Y\in\mathcal{S}^n_+:\langle C,Y\rangle=0\}$ which by the form of C implies that $F\cong T(F)\cong\mathcal{S}^r_+\oplus O_{n-r}$. Lastly, as T is linear, it is a continuous map and since D lies in the relative interior of $T(F)$ it follows that $A=T^{-1}(D)$ is a relative interior point of F. This shows that $F=F_{\mathcal{S}^n_+}(A)$ and the proof is completed. $\square$

Using Theorem 2.3.4 we conclude that every face of the psd cone is exposed.

2.3.5 Corollary. *The cone $\mathcal{S}^n_+$ is facially exposed.*

Proof. Let F be a face of $\mathcal{S}^n_+$ and let $F=F_{\mathcal{S}^n_+}(A)$ for some $A\in\operatorname{relint}F$. By Theorem 2.3.4 it follows that $F=\{X\in\mathcal{S}^n_+:\operatorname{Ker}A\subseteq\operatorname{Ker}X\}$. Let $u_1,\dots,u_k$ be an orthonormal basis for $\operatorname{Ker}A$ and notice that $X\in F$ if and only if $\langle\sum_{i=1}^k u_iu_i^{\mathsf{T}},X\rangle=0$. This implies that F arises as the intersection of $\mathcal{S}^n_+$ with the supporting hyperplane $H=\{X\in\mathcal{S}^n:\langle\sum_{i=1}^k u_iu_i^{\mathsf{T}},X\rangle=0\}$ and thus the claim follows. $\square$

2.3.3 Properties of positive semidefinite matrices

In this section we collect some useful properties of positive semidefinite matrices. The first lemma shows that the Gram decomposition of a positive semidefinite matrix is unique, up to orthogonal transformations.

2.3.6 Lemma. *Consider a matrix $A\in\mathcal{S}^n_+$ and let $A=R_1^{\mathsf{T}}R_1=R_2^{\mathsf{T}}R_2$, where $R_1,R_2\in\mathbb{R}^{d\times n}$. Then, there exists an orthogonal matrix $Q\in O(d)$ such that $R_1=QR_2$.*

Proof. This is a direct consequence of the following well-known geometric fact: Given two sets of vectors $p_1,\dots,p_n\in\mathbb{R}^d$ and $q_1,\dots,q_n\in\mathbb{R}^d$ satisfying $\|p_i-p_j\|=\|q_i-q_j\|$ for all $i\neq j\in[n]$ then there exists an orthogonal matrix $Q\in O(d)$ such that $R_1=QR_2$. $\square$

Another useful observation is that if $X=\operatorname{Gram}(p_1,\dots,p_n)$ then, for any $u\in\mathbb{R}^n$,

$$Xu=0\Longleftrightarrow\sum_{i=1}^n u_ip_i=0.\tag{2.2}$$

Moreover, the following lemma will also be useful.

2.3.7 Lemma. *For any $X, Y \in \mathcal{S}_+^n$ we have that*

$$\langle X, Y \rangle = 0 \Longleftrightarrow XY = 0.$$

The next theorem allows us to deal with positive semidefinite matrices having a block structure.

2.3.8 Theorem. *Consider a symmetric matrix in block form*

$$M = \begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix},$$

where A is positive definite. The matrix $C - B^{\mathsf{T}}A^{-1}B$ is called the Shur complement *of A in M. The following are equivalent:*

(i) M is positive (semi)definite.

(ii) $C - B^{\mathsf{T}}A^{-1}B$ is positive (semi)definite.

Proof. This result follows easily after observing that

$$\begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix} \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C - B^{\mathsf{T}}A^{-1}B \end{pmatrix}.$$

Since

$$\begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix},$$

the equality above is a congruence and the claim follows from Sylvester's law of Inertia [61]. □

The following simple lemma, which is a special case of the *column inclusion property* for psd matrices, is the crucial ingredient for the construction of partial matrices with a unique psd completion.

2.3.9 Lemma. *Consider a symmetric matrix in block form $M = \begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix}$. If M is positive semidefinite then we have that* $\mathrm{Ker}\, A \subseteq \mathrm{Ker}\, B^{\mathsf{T}}$.

Proof. Consider a vector $x \in \mathrm{Ker} A$ and set $z = (x\ 0)^{\mathsf{T}}$. Then $z^{\mathsf{T}}Mz = 0$ and as M is psd it follows that $Mz = 0$. which implies the claim. □

We continue with another simple but useful property.

2.3.10 Lemma. *The block matrix $\begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix}$ is positive semidefinite if and only if the matrix $\begin{pmatrix} A & -B \\ -B^{\mathsf{T}} & C \end{pmatrix}$ is positive semidefinite.*

We conclude this section with a well-known lemma concerning common psd completions of positive semidefinite matrices that will be used numerous times throughout this thesis.

2.3.11 Lemma. *Consider two psd matrices X_i indexed respectively by V_i $(i = 1, 2)$ such that $X_1[V_1 \cap V_2] = X_2[V_1 \cap V_2]$. Then X_1 and X_2 admit a common psd completion X indexed by $V_1 \cup V_2$ with rank equal to* $\max\{\mathrm{rank}(X_1), \mathrm{rank}(X_2)\}$.

Proof. Let $u_j^{(i)}$ ($j \in V_i$) be a Gram representation of X_i ($i = 1, 2$) and assume without loss of generality that the two families of vectors lie in the same space $\mathbb{R}^n$. By Lemma 2.3.6 there exists an orthogonal matrix $Q \in O(n)$ mapping $u_j^{(1)}$ to $u_j^{(2)}$ for $j \in V_1 \cap V_2$. Then, the Gram matrix of the vectors of $Qu_j^{(1)}$ ($j \in V_1$) together with $u_j^{(2)}$ ($j \in V_2 \setminus V_1$) is a common psd completion with the desired properties. □

3
Semidefinite Programming

A semidefinite program is a convex program defined as the minimization of a linear function over an affine section of the cone of positive semidefinite matrices. Semidefinite programming is a far reaching generalization of linear programming that has a powerful duality theory and for which there exist efficient algorithms both in theory and in practice for solving them. In this section we recall all the necessary definitions and background material concerning semidefinite programs that will be used throughout this thesis. Our notation and exposition closely follows [7]. Some other excellent sources include [99, 39, 92].

3.1 Definitions and basic properties

A semidefinite program in canonical primal form is given by:

$$p^* = \sup_X \left\{ \langle C, X \rangle : X \succeq 0,\ \langle A_i, X \rangle = b_i\ (i \in I),\ \langle A_i, X \rangle \leq b_i\ (i \in J) \right\}. \tag{P}$$

Here, $C \in \mathscr{S}^n, A_i \in \mathscr{S}^n$ $(i \in I \cup J)$ and $b \in \mathbb{R}^{|I|+|J|}$ are given and $I \cap J = \emptyset$. Although it is customary to define semidefinite programs involving only linear equalities ($J = \emptyset$), we allow linear inequalities, as they will be useful for later sections. The set

$$\mathscr{P} = \left\{ X \succeq 0,\ \langle A_i, X \rangle = b_i\ (i \in I),\ \langle A_i, X \rangle \leq b_i\ (i \in J) \right\} \tag{3.1}$$

is called the *primal feasible region* and any matrix $X \in \mathscr{P}$ is called *primal feasible*. The program (P) is called *infeasible* if $\mathscr{P} = \emptyset$ and *strictly feasible* if there exists $X \in \mathscr{P}$ such that $X \succ 0$. Additionally, the program (P) is called *rational* if $C \in \mathbb{Q}^{n \times n}$, $A_i \in \mathbb{Q}^{n \times n}$ $(i \in I \cup J)$ and $b \in \mathbb{Q}^{|I|+|J|}$. Sets of the form (3.1) are called SDP-*representable* and in the special case that $J = \emptyset$ they are called *spectrahedra*.

The Lagrangian dual problem of (P) is given by:

$$d^* = \inf_{y,Z} \left\{ \sum_{i \in I \cup J} b_i y_i \ :\ \sum_{i \in I \cup J} y_i A_i - C = Z \succeq 0,\ y_i \geq 0\ (i \in J) \right\}. \tag{D}$$

We denote by $\mathscr{D}$ the set of dual feasible solutions. An expression of the form $\sum_{i\in I} y_i A_i - C \succeq 0$ is called a *Linear Matrix Inequality*.

By construction of the Lagrangian dual of a convex program, the value of any dual feasible solution is an upper bound on p^* and thus d^* is equal to the best (meaning smallest) such bound. This is formalized in the following theorem.

3.1.1 Theorem (Weak duality). *Let $X,(y,Z)$ be a pair of primal-dual feasible solutions for* (P) *and* (D)*, respectively. Then,*

$$\langle C,X\rangle \le b^\mathsf{T} y \ \textit{ and thus } \ p^* \le d^*.$$

Proof. The claim follows directly from the following calculation:

$$b^\mathsf{T} y - \langle C,X\rangle =$$
$$b^\mathsf{T} y - \langle \sum_{i\in I\cup J} y_i A_i - Z, X\rangle = \sum_{i\in J} y_i(b_i - \langle A_i,X\rangle) + \langle Z,X\rangle \ge 0.$$

□

The weak duality theorem has many important consequences. As a first example, weak duality implies that if (D) is unbounded ($d^* = -\infty$) then the primal program is infeasible. Another consequence gives us a simple way to verify the optimality of a pair of primal-dual feasible solutions.

3.1.2 Theorem (Optimality condition). *Let $X,(y,Z)$ be a pair of primal-dual feasible solutions for* (P) *and* (D)*, respectively. If $b^\mathsf{T} y = \langle C,X\rangle$ then $p^* = d^*$ and moreover p^* is attained at X and d^* is attained at (y,Z).*

Proof. Weak duality gives that $\langle C,X\rangle \le p^* \le d^* \le b^\mathsf{T} y$ and using the hypothesis we see that equality holds throughout. □

For a primal feasible matrix $X \in \mathscr{P}$, we denote by J_X the set of inequality constraints that are active at X, i.e.,

$$J_X = \{i \in J : \langle A_i, X\rangle = b_i\}. \tag{3.2}$$

Similarly, for a dual feasible matrix $Z \in \mathscr{D}$ we set

$$J_Z = \{i \in J : y_i > 0\}. \tag{3.3}$$

The next theorem gives conditions that guarantee the optimality of a pair of primal-dual feasible solutions.

3.1.3 Theorem (Complementary Slackness). *Let $X,(y,Z)$ be a pair of primal-dual feasible solutions for* (P) *and* (D)*, respectively. Under the assumption that $p^* = d^*$ we have that $X,(y,Z)$ are primal-dual optimal if and only if $\langle X,Z\rangle = 0$ and $J_Z \subseteq J_X$.*

Proof. The claim follows since for pair of primal-dual feasible solutions $X,(y,Z)$ we have that $\langle C,X\rangle = b^\mathsf{T} y$ if and only if $\langle X,Z\rangle = 0$ and $J_Z \subseteq J_X$. □

The next example illustrates that, unlike linear programming, in the case of semidefinite programming it can happen that $p^* < d^*$. The difference $d^* - p^*$ is called the *duality gap* and we say that *perfect duality* holds if $p^* = d^*$.

3.1.1 Example. *Consider the semidefinite program*

$$\inf\left\{x : \begin{pmatrix} 0 & x & 0 \\ x & y & 0 \\ 0 & 0 & x+1 \end{pmatrix} \succeq 0\right\}.$$

Every feasible solution satisfies $x = 0$ *and thus* $\mathscr{P} = \{(0, y) : y \geq 0\}$ *and* $p^* = 0$*. The dual problem reads:*

$$\sup\left\{-1 : \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ b & 0 & 1 \end{pmatrix} \succeq 0\right\}.$$

Then $\mathscr{D} = \{(a, b) : a \geq b^2\}$ *and* $d^* = -1$*.*

The preceding example raises the question of identifying appropriate conditions under which perfect duality holds for a pair of primal-dual semidefinite programs. The next theorem shows that this can achieved under some mild assumptions.

3.1.4 Theorem. *Consider a pair of primal-dual semidefinite programs as in* (P) *and* (D)*. Assume that* $d^* > -\infty$ *(resp.* $p^* < \infty$*) and that* (D) *(resp.* (P)*) is strictly feasible. Then* $p^* = d^*$ *and moreover the primal (resp. dual) optimal value is attained.*

For a proof of this fact see [39, Theorem 2.2].

A theorem of the alternatives is a statement saying that of two given systems, exactly one of them has a solution. A well known example is Farkas' lemma for linear programming whose geometric interpretation is that either a vector belongs to a given closed convex cone, or there exists a hyperplane separating the vector from the cone. We conclude this section with a theorem of alternatives in the setting of semidefinite programming.

3.1.5 Lemma. *Let* $b \in \mathbb{R}^m$ *and let* $A_1, \ldots, A_m \in \mathscr{S}^n$ *be given. Then exactly one of the following two assertions holds:*

(i) Either there exists $X \in \mathscr{S}^n_{++}$ *such that* $\langle A_j, X\rangle = b_j$ *for* $j = 1, \ldots, m$*.*

(ii) Or there exists a vector $y \in \mathbb{R}^m$ *such that* $\Omega = \sum_{j=1}^m y_j A_j \succeq 0$*,* $\Omega \neq 0$ *and* $b^T y \leq 0$*.*

Moreover, for any $X \succeq 0$ *satisfying* $\langle A_j, X\rangle = b_j$ *(*$j \in [m]$*), we have in (ii)* $\langle X, \Omega\rangle = b^T y = 0$ *and thus* $X\Omega = 0$*.*

Proof. Assume first that both $(i), (ii)$ hold. Then, $\langle X, \Omega\rangle \geq 0$ since $X, \Omega \succeq 0$, and $\langle X, \Omega\rangle = \sum_j b_j y_j \leq 0$; this implies $\langle X, \Omega\rangle = 0$, which contradicts the assumption that $X \succ 0$, $\Omega \succeq 0$ and $\Omega \neq 0$.

Assume that (i) does not hold, i.e., $\mathscr{S}^n_{++} \cap \mathscr{L} = \emptyset$, where $\mathscr{L}$ denotes the affine space $\{X \in \mathscr{S}^n : \langle A_j, X\rangle = b_j \ \forall j\}$. Then, using the separation theorem for convex sets, there exists a hyperplane separating $\mathscr{S}^n_{++}$ and $\mathscr{L}$, i.e., there exists a nonzero matrix $\Omega \in \mathscr{S}^n$ and $\alpha \in \mathbb{R}$ such that $\langle \Omega, X\rangle \geq \alpha$ for all $X \in \mathscr{S}^n_{++}$ and $\langle \Omega, X\rangle \leq \alpha$ for all $X \in \mathscr{L}$. This implies $\Omega \succeq 0$, $\Omega \in \mathscr{L}^\perp$ and $\alpha \leq 0$, and thus (ii) holds. □

3.2 Spectrahedra

In this section we recall some basic geometric properties of the convex sets that arise as the feasible regions of semidefinite programs, known as *spectrahedra*. Additionally we study in detail the geometric properties of the elliptope, a spectrahedron that has received significant amount of attention in the literature.

3.2.1 Basic properties

Formally, a spectrahedron is any convex set obtained as the intersection of the cone of positive semidefinite matrices with an affine subspace. Recently, there has been a surge of interest in the study of spectrahedra due to their relevance to optimization and convex algebraic geometry [27].

The first theorem in this section gives an explicit characterization of the space of perturbations of an element of a spectrahedron (in the more general setting where we allow inequalities).

3.2.1 Theorem. *[84, 44] Let A_i $(i \in I \cup J)$ be a set of n-by-n symmetric matrices and let $b = (b_i) \in \mathbb{R}^{|I|+|J|}$. Consider the convex set*

$$\mathscr{P} = \left\{X \succeq 0 : \langle A_i, X\rangle = b_i \ (i \in I),\ \langle A_i, X\rangle \le b_i \ (i \in J)\right\}. \tag{3.4}$$

Let $X \in \mathscr{P}$, written as $X = PP^{\mathsf{T}}$, where $P \in \mathbb{R}^{n\times r}$ and $r = \operatorname{rank} X$. Then,

$$\operatorname{Pert}_{\mathscr{P}}(X) = \left\{PRP^{\mathsf{T}} : R \in \mathscr{S}^r,\ \langle PRP^{\mathsf{T}}, A_i\rangle = 0 \ (i \in I \cup J_X)\right\}, \tag{3.5}$$

where $J_X = \{i \in J : \langle A_i, X\rangle = b_i\}$. Moreover,

$$\dim F_{\mathscr{P}}(X) = \binom{r+1}{2} - \dim\langle P^{\mathsf{T}} A_i P : i \in I \cup J_X\rangle. \tag{3.6}$$

Proof. Let $Y = PRP^{\mathsf{T}}$ where $R \in \mathscr{S}^r$ and $\langle A_i, Y\rangle = 0$ for all $i \in I \cup J_X$. The matrix $X \pm \lambda Y = P(I \pm \lambda R)P^{\mathsf{T}}$ is clearly positive semidefinite for small enough $\lambda > 0$ and moreover $\langle A_i, X \pm \lambda Y\rangle = \langle A_i, X\rangle = b_i$ for all $i \in I \cup J_X$. This implies that $Y \in \operatorname{Pert}_{\mathscr{P}}(X)$.

For the other direction, let $Y \in \operatorname{Pert}_{\mathscr{P}}(X)$. By definition, there exists $\lambda > 0$ such that $X \pm \lambda Y \in \mathscr{P}$ and thus $\langle A_i, Y\rangle = 0$ for all $i \in I \cup J_X$. Next we complete the matrix P to a non-singular matrix $\widetilde{P}$ and set $C = \widetilde{P}^{-1} Y (\widetilde{P}^{-1})^{\mathsf{T}}$. Then,

$$X \pm \lambda Y = \widetilde{P}\left(\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \pm \lambda \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\right)\widetilde{P}^{\mathsf{T}},$$

and since $X \pm \lambda Y \in \mathscr{P}$ and $\widetilde{P}$ is invertible it follows that

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \pm \lambda \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \succeq 0.$$

As $\lambda > 0$, the diagonal entries of C_{22} need to be zero which in turn implies that $C_{22} = C_{12} = C_{21} = 0$. Then we obtain that $Y = \widetilde{P} C \widetilde{P}^{\mathsf{T}} = P C_{11} P^{\mathsf{T}}$ and the claim follows.

Lastly, to show (3.6) we use the fact that $\dim F_{\mathscr{P}}(X) = \dim \operatorname{Pert}_{\mathscr{P}}(X)$. By (3.5) we see that $\dim \operatorname{Pert}_{\mathscr{P}}(X)$ is equal to the dimension of the orthogonal complement of $\{P^{\mathsf{T}} A_i P : i \in I \cup J_X\}$ in the space $\mathscr{S}^r$ which implies the claim. □

As a direct application we obtain the general bound (1.3) for the existence of bounded-rank elements of spectrahedra.

3.2.2 Corollary. *Consider a spectahedron of the form*

$$\mathscr{P} = \{X \succeq 0 : \langle A_i, X\rangle = b_i \ (i \in [m])\}.$$

If $\mathscr{P} \neq \emptyset$ then it has a feasible solution of rank at most

$$\left\lfloor \frac{\sqrt{8m+1}-1}{2} \right\rfloor.$$

Proof. As $\mathscr{P}$ is a nonempty closed convex set which does not contain straight lines it has an extreme point; cf. [24, §2.3, Lemma 3.5]. Call this extreme point X and let $r = \operatorname{rank} X$. As $X \in \operatorname{ext} \mathscr{P}$ we have that $\dim F_{\mathscr{P}}(X) = 0$ which combined with (3.6) implies that $\binom{r+1}{2} \leq m$ and thus $r \leq \left\lfloor \frac{\sqrt{8m+1}-1}{2} \right\rfloor$. □

As a second application of Theorem 3.2.1, we obtain the following characterization for the extreme points of $\mathscr{P}$ that will be useful for later chapters.

3.2.3 Corollary. *Consider a matrix $X \in \mathscr{P}$ (as in (3.4)), written as $X = PP^{\mathsf{T}}$, where $P \in \mathbb{R}^{n\times r}$ and $r = \operatorname{rank} X$. The following assertions are equivalent:*

(i) X is an extreme point of $\mathscr{P}$.

(ii) If $R \in \mathscr{S}^r$ satisfies $\langle P^{\mathsf{T}} A_i P, R\rangle = 0$ for all $i \in I \cup J_X$, then $R = 0$.

(iii) $\operatorname{lin}\{P^{\mathsf{T}} A_i P : i \in I \cup J_X\} = \mathscr{S}^r$.

Proof. The equivalence (ii) $\Longleftrightarrow$ (iii) is immediate and the equivalence (i) $\Longleftrightarrow$ (iii) follows from (3.5). □

3.2.2 The elliptope

In this section we introduce one of the most extensively studied spectrahedra and investigate its geometric properties. This spectrahedron arises as the feasible region of the semidefinite relaxation for MAX CUT, introduced by Goemans and Williamson [51].

3.2.4 Definition. *The n-dimensional* elliptope*, denoted by $\mathscr{E}_n$, is the set of n-by-n positive semidefinite matrices whose diagonal elements are all equal to one.*

The 3-dimensional elliptope $\mathscr{E}_3$ (or rather, its bijective image in $\mathbb{R}^3$ obtained by considering only the upper triangular part of matrices in $\mathscr{E}_3$) is illustrated in Figure 3.1.

Positive semidefinite matrices whose diagonal entries are all equal to one are also known as *correlation matrices*. We now briefly explain where this name is originating from. Recall that for a random variable X, we denote by $\mathbb{E}(X)$ its mean value and by and $\mathbb{V}(X)$ its variance. Given two random variables X, Y their *covariance*, denoted by $\operatorname{cov}(X,Y)$, is defined as

$$\operatorname{cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

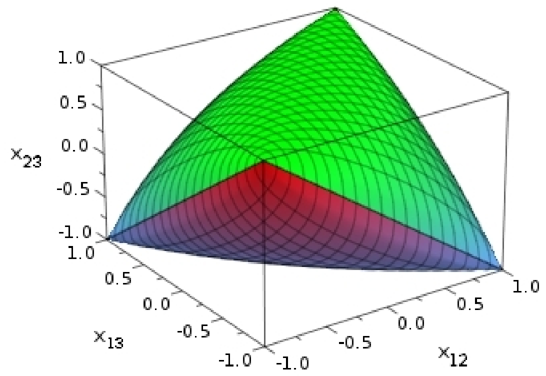

Figure 3.1: The elliptope $\mathscr{E}_3$.

Assuming that $\mathbb{V}(X), \mathbb{V}(Y) \neq 0$, their *correlation*, denoted by $\mathrm{cor}(X,Y)$, is defined as

$$\mathrm{cor}(X,Y) = \mathrm{cov}\left(X/\sqrt{\mathbb{V}(X)}, Y/\sqrt{\mathbb{V}(Y)}\right).$$

The *correlation matrix* of a family of real valued random variables $X_1,\dots,X_n$, denoted by $\mathrm{cor}(X_1,\dots,X_n)$, is the n-by-n matrix whose (i,j) entry is given by $\mathrm{cor}(X_i,X_j)$.

We now show that for any family of random variables $X_1,\dots,X_n$ the matrix $\mathrm{cor}(X_1,\dots,X_n)$ is an element of the elliptope $\mathscr{E}_n$. Indeed, using the well-known property $\mathbb{V}(\alpha X+\beta Y) = \alpha^2\mathbb{V}(X)+\beta^2\mathbb{V}(Y)+2\alpha\beta\mathrm{cov}(X,Y)$, it follows that

$$x^{\mathsf{T}}\mathrm{cor}(X_1,\dots,X_n)x = \mathbb{V}\left(\sum_{i=1}^n x_i \frac{X_i}{\sqrt{\mathbb{V}(X_i)}}\right),$$

and thus $\mathrm{cor}(X_1,\dots,X_n)$ is positive semidefinite (recall that the variance of a random variable X is equal to $\mathbb{E}(X-\mathbb{E}X)^2$ and thus it is always nonnegative.) Lastly, since $\mathrm{cor}(X,X)=1$, the diagonal entries of $\mathrm{cor}(X_1,\dots,X_n)$ are all equal to one.

Conversely, any element of the elliptope $\mathscr{E}_n$ can be expressed as the correlation matrix of some family of random variables. Indeed, let $X \in \mathscr{E}_n$ and consider a family of randon variables $X_1,\dots,X_n$ such that $\mathrm{cor}(X_1,\dots,X_n) = I_n$. Then,

$$\mathrm{cor}(\sum_{j=1}^n X_{1j}^{1/2}X_j,\dots,\sum_{j=1}^n X_{nj}^{1/2}X_j) = X^{1/2}\mathrm{cor}(X_1,\dots,X_n)X^{1/2} = X.$$

We now recall some basic facts concerning the facial structure of the elliptope that will be relevant for this thesis. Clearly, the only face of the convex set $\{X \in \mathscr{S}^n : X_{ii}=1\ (i\in[n])\}$ is the set itself. As $\mathscr{E}_n = \mathscr{S}^n_+ \cap \{X \in \mathscr{S}^n : X_{ii} = 1\ (i \in [n]\}$, Theorem 2.3.4 yields the following characterization for the faces of $\mathscr{E}_n$.

3.2.5 Lemma. *For a matrix $X \in \mathscr{E}_n$, the smallest face of $\mathscr{E}_n$ containing X is given by*

$$F_{\mathscr{E}_n}(X) = \{Y \in \mathscr{E}_n : \mathrm{Ker}\,X \subseteq \mathrm{Ker}\,Y\}. \tag{3.7}$$

It follows from (3.7) that two matrices in the relative interior of a face F of $\mathcal{E}_n$ have the same rank, while $\operatorname{rank} X > \operatorname{rank} Y$ if X is in the relative interior of F and Y lies on the boundary of F.

The next proposition gives an explicit description of the space of perturbations of a matrix $X \in \mathcal{E}_n$ and its proof is a direct application of Theorem 3.2.1.

3.2.6 Proposition. *Consider a matrix $X \in \mathcal{E}_n$ with $\operatorname{rank} X = r$ and let $u_1, \dots, u_n \in \mathbb{R}^r$ be a Gram representation of X. Moreover, let P be the $n \times r$ matrix with rows $u_1, \dots, u_n$ and set $\mathcal{U}_V = \langle u_1 u_1^\mathsf{T}, \dots, u_n u_n^\mathsf{T} \rangle \subseteq \mathcal{S}_r$. The space of perturbations at X is given by*

$$\operatorname{Pert}_{\mathcal{E}_n}(X) = \{PRP^\mathsf{T} : R \in \mathcal{S}_r,\ \langle R, u_i u_i^\mathsf{T} \rangle = 0\ (i \in [n])\} \tag{3.8}$$

and the dimension of the smallest face of $\mathcal{E}_n$ containing X is

$$\dim F_{\mathcal{E}_n}(X) = \binom{r+1}{2} - \dim\ \mathcal{U}_V. \tag{3.9}$$

In particular, X is an extreme point of $\mathcal{E}_n$ if and only if

$$\binom{r+1}{2} = \dim\ \mathcal{U}_V. \tag{3.10}$$

Hence, if $X \in \operatorname{ext} \mathcal{E}_n$ with $\operatorname{rank} X = r$ then

$$\binom{r+1}{2} \leq n. \tag{3.11}$$

The next theorem shows that every number in the range prescribed in (3.11) corresponds to an extremal element of $\mathcal{E}_n$.

3.2.7 Theorem. *[84] For any natural number r satisfying $\binom{r+1}{2} \leq n$ there exists a matrix $X \in \mathcal{E}_n$ which is an extreme point of $\mathcal{E}_n$ and has rank equal to r.*

In Chapter 7 it will be useful to have the exact characterization of the extreme points of the elliptope $\mathcal{E}_3$.

3.2.8 Theorem. *[56] A matrix $X = (x_{ij}) \in \mathcal{E}_3$ is an extreme point of $\mathcal{E}_3$ if either $\operatorname{rank} X = 1$, or $\operatorname{rank} X = 2$ and $|x_{ij}| < 1$ for all $i \neq j \in \{1,2,3\}$.*

3.3 Degeneracy in semidefinite programming

In this section we go back to the primal-dual pair of semidefinite programs introduced in Section 3.1. Our main goal in this section is to introduce the concept of (non)degeneracy in semidefinite programming and state some basic theorems guaranteeing the existence of a unique optimal solution to a semidefinite program. These conditions will play a crucial role in Chapter 11.

3.3.1 Definition. *Let $X, (y, Z)$ be a pair of primal-dual optimal solutions for* (P) *and* (D)*, respectively. The solutions $X, (y, Z)$ are called* complementary *if $XZ = 0$ and* strict complementary *if moreover $\operatorname{rank} X + \operatorname{rank} Z = n$.*

We denote by $\mathcal{R}_r$ the manifold of symmetric n-by-n matrices with rank equal to r. Consider a matrix $X \in \mathcal{R}_r$ and let $X = Q\Lambda Q^\mathsf{T}$ be its spectral decomposition, where Q is an orthogonal matrix whose columns are the eigenvectors of X and Λ is the diagonal matrix with the corresponding eigenvalues as diagonal entries. Without loss of generality we may assume that $\Lambda_{ii} \neq 0$ for $i \in [r]$.

3.3.2 Theorem. *[12, 121] The* tangent space *of $\mathscr{R}_r$ at X is given by*

$$\mathscr{T}_X = \left\{ Q \begin{pmatrix} U & V \\ V^\mathsf{T} & 0 \end{pmatrix} Q^\mathsf{T} : U \in \mathscr{S}^r, V \in \mathbb{R}^{r\times(n-r)} \right\}. \tag{3.12}$$

Hence, its orthogonal complement is defined by

$$\mathscr{T}_X^\perp = \left\{ Q \begin{pmatrix} 0 & 0 \\ 0 & W \end{pmatrix} Q^\mathsf{T} : W \in \mathscr{S}^{n-r} \right\}. \tag{3.13}$$

We will also use the equivalent description:

$$\mathscr{T}_X^\perp = \{M \in \mathscr{S}^n : XM = 0\}. \tag{3.14}$$

We now introduce the notions of nondegeneracy and strict complementarity for a pair of primal-dual semidefinite programs (P) and (D) in standard form.

3.3.3 Definition. *[7] Consider the pair of primal and dual semidefinite programs (P) and (D). A matrix $X \in \mathscr{P}$ is called* primal nondegenerate *if*

$$\mathscr{T}_X + \operatorname{lin}\{A_i : i \in I \cup J_X\}^\perp = \mathscr{S}^n. \tag{3.15}$$

The pair $(y, Z) \in \mathscr{D}$ is called dual nondegenerate *if*

$$\mathscr{T}_Z + \operatorname{lin}\{A_i : i \in I \cup J_Z\} = \mathscr{S}^n. \tag{3.16}$$

Recall that J_X (resp. J_Z) denotes the set of constraints that are active at X; cf. (3.2) (resp. (3.3)).

Next we present some well known results that provide necessary and sufficient conditions for the unicity of optimal solutions in terms of the notions of primal or dual nondegeneracy and strict complementarity. With the intention to make the section self-contained we have also included short proofs.

3.3.4 Theorem. *[7] Assume that the optimal values of* (P) *and* (D) *are equal and that both are attained. If* (P) *has a nondegenerate optimal solution, then* (D) *has a unique optimal solution. (Analogously, if* (D) *has a nondegenerate optimal solution, then* (P) *has a unique optimal solution.)*

Proof. Let X be a nondegenerate optimal solution of (P) and let $(y^{(1)}, Z_1)$, $(y^{(2)}, Z_2)$ be two dual optimal solutions. Complementary slackness implies that $y_j^{(1)} = y_j^{(2)} = 0$ holds for every $i \in J \setminus J_X$. Hence, $Z_1 - Z_2 \in \operatorname{lin}\{A_i : i \in I \cup J_X\}$. As there is no duality gap we have that $XZ_1 = XZ_2 = 0$ and then (3.14) implies that $Z_1 - Z_2 \in \mathscr{T}_X^\perp$. These two facts combined with the assumption that X is primal nondegenerate imply that $Z_1 = Z_2$. The other case is similar. □

We continue with a simple observation that will be useful for the remainder of this section. Let $X, (y, Z)$ be a pair of strict complementary solutions. By assumption, $ZX = XZ = 0$ which implies that X and Z can be simultaneously diagonalized, i.e., there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $X = Q\Lambda_1 Q^\mathsf{T}$ and $Z = Q\Lambda_2 Q^\mathsf{T}$. Let $r = \operatorname{rank} X$. As $XZ = 0$ it follows that $\Lambda_1\Lambda_2 = 0$ and since $\operatorname{rank} X + \operatorname{rank} Z = n$ we obtain that

$$X = Q \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} Q^\mathsf{T} = Q_1\Lambda_1 Q_1^\mathsf{T}, \; Z = Q \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} Q^\mathsf{T} = Q_2\Lambda_2 Q_2^\mathsf{T}, \tag{3.17}$$

where Λ_1 and Λ_2 are diagonal matrices of sizes r and $n-r$, respectively.

The next lemma provides a characterization of the space of perturbations in terms of tangent spaces for a pair of strict complementary optimal solutions.

3.3.5 Lemma. *Assume that the optimal values of* (P) *and* (D) *are equal and that both are attained. Let $X, (y, Z)$ be a strict complementary pair of primal and dual optimal solutions for* (P) *and* (D)*, respectively. Then,*

$$\mathrm{Pert}_{\mathscr{P}}(X) = \mathrm{lin}\{A_i : i \in I \cup J_X\}^{\perp} \cap \mathscr{T}_Z^{\perp}, \tag{3.18}$$

$$\mathrm{Pert}_{\mathscr{D}}(Z) = \mathrm{lin}\{A_i : i \in I \cup J_Z\}^{\perp} \cap \mathscr{T}_X^{\perp}. \tag{3.19}$$

Proof. By assumption the matrices X and Z can be simultaneously diagonalized by an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$. Let $s = \mathrm{rank}\, Z$ and $r = \mathrm{rank} X$. Using (3.17) it follows that

$$\mathscr{T}_Z = \left\{ Q \begin{pmatrix} 0 & V \\ V^{\mathsf{T}} & U \end{pmatrix} Q^{\mathsf{T}} : U \in \mathscr{S}^s, V \in \mathbb{R}^{(n-s)\times s} \right\}$$

and that

$$\mathscr{T}_Z^{\perp} = \left\{ Q \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} Q^{\mathsf{T}} : W \in \mathscr{S}^{n-s} \right\}.$$

Thus we have that $\mathscr{T}_X^{\perp} = \{Q_2 W Q_2^{\mathsf{T}} : W \in \mathscr{S}^{n-r}\}$ and $\mathscr{T}_Z^{\perp} = \{Q_1 W Q_1^{\mathsf{T}} : U \in \mathscr{S}^r\}$ and the claim follows directly from (3.5). □

The next theorem establishes the converse of Theorem 3.3.4, assuming strict complementarity.

3.3.6 Theorem. *[7] Assume that the optimal values of* (P) *and* (D) *are equal and that both are attained. Let $X, (y, Z)$ be a strict complementary pair of optimal solutions for* (P) *and* (D)*, respectively, and assume that $J_X = J_Z$. If X is the unique optimal solution of* (P) *then (y, Z) is dual nondegenerate. (Analogously, if (y, Z) is the unique optimal solution of* (D) *then X is primal nondegenerate.)*

Proof. By assumption, X is the unique optimal solution of (P). Hence X is an extreme point of the primal feasible region and thus, using (3.18), we obtain that $\mathscr{T}_Z + \mathrm{lin}\{A_i : i \in I \cup J_X\} = \mathscr{S}^n$. As $J_X = J_Z$, (3.16) holds and thus (y, Z) is dual nondegenerate. □

The next theorem gives a characterization for the extreme points of $\mathscr{P}$, assuming strict complementarity.

3.3.7 Theorem. *Assume that the optimal values of* (P) *and* (D) *are equal and that both are attained. Let $X, (y, Z)$ be a pair of strict complementary optimal solutions of the primal and dual programs* (P) *and* (D)*, respectively, and assume that $J_X = J_Z$. The following assertions are equivalent:*

(i) X is an extreme point of $\mathscr{P}$.

(ii) X is the unique primal optimal solution of (P).

(iii) Z is a dual nondegenerate.

Proof. The equivalence $(ii) \iff (iii)$ follows directly from Theorems 3.3.4 and 3.3.6 and the equivalence $(i) \iff (iii)$ follows by Lemma 3.3.5 and the definition of dual nondegeneracy from (3.16). □

Note that Theorems 3.3.6 and 3.3.7 still hold if we replace the condition $J_X = J_Z$ by the weaker condition:

$$\forall i \in J_X \setminus J_Z \ \ A_i \in \mathcal{T}_Z + \mathrm{lin}\{A_i : i \in I \cup J_X\}. \tag{3.20}$$

Note also that this condition is automatically satisfied in the case when $J = \emptyset$, i.e., when the semidefinite program (P) involves only linear equations.

3.4 The Strong Arnold Property

In this section we introduce the Strong Arnold Property (SAP), which is used in the definition of the parameter $\nu^{=}(\cdot)$, studied in Section 6.2. Our main result is that for a certain class of SDP's, the conditions of primal and dual nondegeneracy (cf. Section 3.3) are equivalent to the Strong Arnold Property.

We start the discussion with some necessary definitions. With any graph $G = (V = [n], E)$ we associate the linear space

$$\mathcal{L}(G) = \{M \in \mathcal{S}^n : M_{ij} = 0 \ \text{ for all distinct } i, j \in V \text{ with } ij \notin E\}.$$

3.4.1 Definition. *For a graph $G = (V = [n], E)$, a matrix $M \in \mathcal{L}(G)$ is said to satisfy the* Strong Arnold Property *if*

$$\mathcal{T}_M + \mathrm{lin}\{E_{ij} : \{i,j\} \in V \cup E\} = \mathcal{S}^n. \tag{3.21}$$

Recall that $\mathcal{T}_M$ denotes the tangent space at M of the manifold of n-by-n symmetric matrices of rank equal to $\operatorname{rank} M$ (cf. (3.12)).

The SAP has received significant attention due to its relevance to the Colin de Verdière graph parameter $\mu(\cdot)$, introduced and studied in [42]. For a graph $G = ([n], E)$, the parameter $\mu(G)$ is defined as the maximum corank of a symmetric n-by-n matrix M such that: $M_{ij} < 0$ if $ij \in E$, $M_{ij} = 0$ if $ij \notin E$, M has exactly one negative eigenvalue of multiplicity one and M satisfies the Strong Arnold Property.

Colin de Verdière introduced the $\mu(\cdot)$ parameter motivated by the problem of estimating the maximum multiplicity of the second eigenvalue of Schrödinger operators. The parameter $\mu(\cdot)$ is important since it provides an algebraic characterization of many important topological graph properties. Specifically, it is known that:

- $\mu(G) \le 1$ if and only if G is a disjoint union of paths.
- $\mu(G) \le 2$ if and only if G is outerplanar.
- $\mu(G) \le 3$ if and only if G is planar.
- $\mu(G) \le 4$ if and only if G is linklessly embeddable.

Here, the first three items are due to Colin de Verdière [42]. For the fourth item, necessity follows from [116] and sufficiency from [88].

By taking orthogonal complements in (3.21) and using (3.14), we arrive at the following equivalent expression for the SAP:

$$X \in \mathscr{S}^n,\ MX = 0,\ X_{ij} = 0 \text{ for all } \{i,j\} \in V \cup E \Longrightarrow X = 0. \tag{3.22}$$

Our next goal is to give a geometric characterization of matrices satisfying the SAP using the notion of null space representations. Consider a matrix $M \in \mathscr{S}^n$, fix an arbitrary basis for $\operatorname{Ker} M$ and form the n-by-$\operatorname{corank} M$ matrix that has as columns the basis elements. The vectors corresponding to the rows of the resulting matrix form a *nullspace representation* of M. If we impose structure on M in terms of some graph G, nullspace representations of M exhibit intriguing geometric properties and have been extensively studied (see e.g. [89]).

The next theorem shows that null space representations of matrices satisfying the SAP exhibit some interesting geometric properties. The equivalence between the first and the third item in the next theorem was rediscovered independently in [127, Theorem 4.2] and [50, Lemma 3.1].

3.4.2 Theorem. *Consider a graph $G = ([n], E)$ and a matrix $M \in \mathscr{L}(G)$ with* $\operatorname{corank} M = d$. *Let $P \in \mathbb{R}^{n\times d}$ be a matrix whose columns form an orthonormal basis for* $\operatorname{Ker} M$ *and let $\{p_1, \dots, p_n\}$ denote the row vectors of P. The following assertions are equivalent:*

(i) M satisfies the Strong Arnold Property.

(ii) PP^{T} is an extreme point of the spectrahedron

$$\{X \succeq 0 : \langle E_{ij}, X\rangle = p_i^{\mathsf{T}} p_j \text{ for } \{i,j\} \in V \cup E\}.$$

(iii) For any matrix $R \in \mathscr{S}^d$ the following holds:

$$p_i^{\mathsf{T}} R p_j = 0 \text{ for all } \{i,j\} \in V \cup E \Longrightarrow R = 0.$$

Proof. The equivalence $(ii) \Longleftrightarrow (iii)$ follows directly from Corollary 3.2.3.
$(i) \Longrightarrow (iii)$ Let $R \in \mathscr{S}^d$ such that $p_i^{\mathsf{T}} R p_j = 0$, i.e., $\langle PRP^{\mathsf{T}}, E_{ij}\rangle = 0$ for all $\{i,j\} \in V \cup E$. Thus the matrix $Y = PRP^{\mathsf{T}}$ belongs to $\operatorname{lin}\{E_{ij} : \{i,j\} \in V \cup E\}^{\perp}$ and satisfies $MY = 0$. By (3.14) we have that $Y \in \mathscr{T}_M^{\perp}$ and then (i) implies $Y = 0$ and thus $R = 0$ (since $P^{\mathsf{T}} P = I_r$).
$(iii) \Longrightarrow (i)$ Write $M = Q \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} Q^{\mathsf{T}}$, where $Q = (Q_1\ P)$ is orthogonal and the columns of Q_1 form a basis of the range of M. Consider a matrix $Y \in \mathscr{T}_M^{\perp} \cap \operatorname{lin}\{E_{ij} : \{i,j\} \in \overline{E}\}$. Then, by (3.13), $Y = PRP^{\mathsf{T}}$ for some matrix $R \in \mathscr{S}^d$. Moreover, $\langle Y, E_{ij}\rangle = \langle PRP^{\mathsf{T}}, E_{ij}\rangle = 0$ for all $\{i,j\} \in V \cup E$, which by (iii) implies that $R = 0$ and thus $Y = 0$. □

Our final observation in this section is that a psd matrix having the SAP can be also understood as a nondegenerate solution of a certain semidefinite program.

3.4.3 Theorem. *Consider a graph $G = ([n], E)$ and let $M \in \mathscr{L}(G) \cap \mathscr{S}_n^+$. The following assertions are equivalent:*

(i) M satisfies the Strong Arnold Property.

(ii) M is a primal nondegenerate solution for the semidefinite program:

$$\sup_X \{\langle C, X\rangle : \langle E_{ij}, X\rangle = 0 \text{ for } \{i,j\} \in \overline{E},\ X \succeq 0\},$$

for any $C \in \mathscr{S}^n$.

(iii) M is a dual nondegenerate solution for the semidefinite program:

$$\sup_X \{0 : \langle E_{ij}, X\rangle = a_{ij} \text{ for } \{i,j\} \in V \cup E,\ X \succeq 0\}, \tag{3.23}$$

for any $a \in \mathscr{S}_+(G)$.

Proof. Taking orthogonal complements in (3.21) we see that M satisfies the SAP if and only if $\mathscr{T}_M{}^\perp \cap \text{lin}\{E_{ij} : \{i,j\} \in \overline{E}\} = \{0\}$. Moreover, observe that the feasible region of the dual of the semidefinite program (3.23) is equal to $\mathscr{S}^n_+ \cap \mathscr{L}(G)$. Now, using (3.15), we obtain the equivalence of $(i), (ii)$ and (iii). □

This last theorem shows that for certain SDP's, identifying whether a matrix is primal (resp. dual) nondegenerate reduces to checking whether the matrix has the Strong Arnold Property. This observation could prove to be useful, since there is a vast literature concerning the Strong Arnold Property, which could potentially be useful when translated into the framework of semidefinite programming.

3.5 Complexity aspects of semidefinite programming

Our goal in this section is to discuss complexity aspects of semidefinite programming and point out the similarities and differences with linear programming.

It is a fundamental result that if a system $Ax \le b$ of *rational* linear inequalities is feasible then it also has a rational solution whose bit size is polynomially bounded by the bit sizes of A and b [120, Theorem 10.1]. This fact combined with Farkas' lemma for linear programming implies that deciding whether a system of rational linear inequalities is feasible belongs to NP∩ co-NP [120, Corollary 10.1a]. Moreover, it also implies that if a rational linear program $\max\{c^{\mathsf{T}}x : Ax \le b\}$ is feasible then it has a rational optimal solution whose bit size is polynomially bounded in the bit sizes of A, b and c.

In contrast to this, the problem of deciding feasibility of a rational semidefinite program has unknown complexity. The following two examples taken from [76] illustrate two problematic situations. The first example shows that there exist SDP's with rational data that have only irrational solutions. For this, consider the matrix

$$\begin{pmatrix} 2x & 2 & 0 & 0 \\ 2 & x & 0 & 0 \\ 0 & 0 & 2 & x \\ 0 & 0 & x & 1 \end{pmatrix} \tag{3.24}$$

and notice that $x = \sqrt{2}$ is the only value for which this matrix is positive semidefinite. The second example shows that there exist semidefinite programs where all feasible solutions have bit size exponential in the bit size of the data matrices. Indeed, for the semidefinite program

$$\max\left\{x_n : \begin{pmatrix} 1 & 2 \\ 2 & x_1 \end{pmatrix} \succeq 0,\ \begin{pmatrix} 1 & x_{i-1} \\ x_{i-1} & x_i \end{pmatrix} \succeq 0,\ (i = 2, \ldots, n)\right\}, \tag{3.25}$$

it is easy to verify that every feasible solution satisfies $x_n \geq 2^{2^n}$.

The problem of understanding the exact complexity status of semidefinite programming stands out as one of the most important questions in the theory of semidefinite programming. The most important known complexity result, due to Ramana, is that the problem of deciding the feasibility of a rational semidefinite program belongs to NP if and only if it belongs to co-NP [110].

Concerning the complexity of solving a rational semidefinite program, there is no algorithm known that solves *every* SDP in time polynomial in the input size. Indeed, as was already illustrated in the examples given in (3.24) and (3.25), even the representation of the output of a semidefinite program can be problematic in the bit model of computation. Nevertheless, under some suitable and not too restrictive conditions it is possible to devise algorithms that permit us to *approximately* solve SDP's within arbitrary precision in polynomial time. The existence of such an algorithm follows from general results on the ellipsoid method [58].

We now state the main complexity result concerning the solvability of semidefinite programs. Consider a spectrahedron of the form

$$\mathscr{P} = \{X \succeq 0 : \langle A_i, X\rangle = b_i \ (i \in [m])\},$$

where $A_1, \ldots, A_m \in \mathbb{Q}^{n\times n}$ and $b_1, \ldots, b_m \in \mathbb{Q}$. For $\epsilon > 0$ define

$$S(\mathscr{P}, \epsilon) = \mathscr{P} + B(0, \epsilon) \ \text{ and } \ S(\mathscr{P}, -\epsilon) = \{x : B(x, \epsilon) \subseteq \mathscr{P}\},$$

where $B(0, \epsilon)$ denotes the Euclidean ball with respect to the Frobenious norm. Assume that there exists an integer R known a priori with the property that either $\mathscr{P} = \emptyset$ or $\mathscr{P} \cap B(0, R) \neq 0$. Then, there exists an algorithm that solves the "weak optimization problem" over $\mathscr{P}$ whose running time is polynomial in $n, m, \log R, \log \frac{1}{\epsilon}$ and the bit size of the matrices $(A_i)_{i=1}^m$ and the scalars $(b_i)_{i=1}^m$.

Recall that the weak optimization problem over $\mathscr{P}$ is defined as follows (cf. [58, Problem 2.1.10]): For any rational matrix $C \in \mathbb{Q}^{n\times n}$ and rational $\epsilon > 0$ either

- find a matrix $X^* \in \mathbb{Q}^{n\times n}$ such that $X^* \in S(\mathscr{P}, \epsilon)$ and $\langle C, X\rangle \leq \langle C, X^*\rangle + \epsilon$ for every $X \in S(\mathscr{P}, -\epsilon)$ or
- assert that $S(\mathscr{P}, -\epsilon)$ is empty.

It is important to realize that the complexity result presented above does not imply the polynomial-time solvability of an arbitrary semidefinite program because the size of R can be large. Indeed, the semidefinite program given in (3.25) is such an example, since every feasible solution has size exponential in n. This complexity result will guarantee polynomial-time solvability only when we can provide good bounds for the size of R. Luckily, this is very often the case for applications.

Another important point is that the ellipsoid method is the *only known* method that (under suitable assumptions) allows us to prove the polynomial time solvability of semidefinite programs (within arbitrary precision). Indeed, for example for *interior-point* algorithms there are no known polynomial bounds on the bit size of the numbers occurring during the execution of these algorithms. For more details the reader is referred to [111, §9.3.1] and [92, §2.6].

On the other hand the performance of the ellipsoid algorithm is poor in practice and the tool of choice for solving semidefinite programs is interior point algorithms. The development of interior-point algorithms for semidefinite programs was pioneered independently by Nesterov and Nemirovski [95] and Alizadeh [6].

Lastly, Porkolab and Khachiyan showed that deciding feasibility of a rational semidefinite program can be done in polynomial time if either n (i.e., the size of the matrices) or m (i.e., the number of hyperplanes) is a fixed constant [109].

4

The cut polytope and its relatives

In this chapter we introduce and give some basic properties of the cut polytope, the metric polytope, and the elliptope of a graph. The cut polytope of a graph, defined as the convex hull of the incidence vectors of all cuts in the graph, arises naturally in a number of disparate fields ranging from combinatorial optimization to quantum information theory. As our understanding of the cut polytope is limited, there has been a significant amount of work in identifying tractable relaxations for the cut polytope. In this chapter we introduce two such relaxations: The first one is a polyhedral relaxation known as the metric polytope of the graph. The second one is a non-polyhedral relaxation known as the elliptope of the graph. For a comprehensive treatment of this material the reader is referred to [44].

4.1 The cut polytope

4.1.1 Definition. *For graph $G = (V, E)$ and $S \subseteq V$, the* cut vector *defined by S, denoted by $\delta_G(S) \in \mathbb{R}^E$, is defined as*

$$\delta_G(S)_{ij} = \begin{cases} 1 & \text{if } |S \cap \{i, j\}| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The cut polytope *of a graph $G = (V, E)$, denoted by $\mathrm{CUT}^{01}(G)$, is defined as the convex hull of the cut vectors $\delta_G(S)$ for all subsets $S \subseteq V$. For convenience, we also denote by $\delta_G(S)$ the set of edges in G that cross the cut defined by $S \subseteq V$.*

It will be sometimes more convenient to work with ± 1 variables, rather than $0, 1$ variables. Formally, consider the linear map:

$$f : \mathbb{R}^E \mapsto \mathbb{R}^E \qquad x \mapsto e - 2x, \tag{4.1}$$

where $e \in \mathbb{R}^E$ denotes the all ones vector. Then, the cut polytope in ± 1 variables is given by

$$\mathrm{CUT}^{\pm 1}(G) = f(\mathrm{CUT}^{01}(G)).$$

Moreover, we denote by $\mathrm{CUT}_n^{\pm 1}$ the set of n-dimensional cut matrices, i.e.,

$$\mathrm{CUT}_n^{\pm 1} = \mathrm{conv}\{xx^\top : x \in \{\pm 1\}^n\}.$$

The following property of the cut polytope in ± 1 variables will be useful in later chapters.

4.1.2 Lemma. *Consider a graph $G = ([n], E)$. Then*

$$\mathrm{CUT}^{\pm 1}(G) = \pi_E(\mathrm{CUT}_n^{\pm 1}).$$

To ease notation, whenever it is clear (or irrelevant) if we are working in the ± 1 or the $0, 1$ setting we will drop superscripts and just write $\mathrm{CUT}(G)$.

The study of the cut polytope $\mathrm{CUT}^{01}(G)$ is largely motivated by its relevance to the MAX CUT problem in combinatorial optimization.

4.1.3 Definition. *For a graph $G = (V, E)$ with edge weights $w \in \mathbb{R}^E$ the* MAX CUT *problem asks for a cut $\delta_G(S)$ for which $\sum_{ij \in \delta_G(S)} w_{ij}$ is maximized.*

The decision version of the MAX CUT problem is one of the first problems that was shown to be NP-complete [93]. Moreover, deciding whether a rational vector $x \in \mathbb{Q}^E$ belongs to the cut polytope is also an NP-complete problem [15].

Clearly, the MAX CUT problem can be formulated as a linear programming problem over the cut polytope as follows:

$$\mathrm{mc}(G, w) = \max\{w^\top x : x \in \mathrm{CUT}^{01}(G)\}.$$

This reformulation renders the problem amenable to linear programming techniques provided that the linear inequality description for the cut polytope is available. Unfortunately, it is known that there is no computationally tractable linear inequality description of a polyhedron associated with an NP-complete problem unless NP = co-NP [64].

The cut polytope admits a very important symmetric transformation which we introduce below.

4.1.4 Definition. *Given a vector $w \in \mathbb{R}^E$ and $S \subseteq V$ consider a new vector $w^{\delta_G(S)} \in \mathbb{R}^E$ defined as*

$$w_e^{\delta_G(S)} = \begin{cases} -w_e & \text{if } e \in \delta_G(S) \\ w_e & \text{otherwise,} \end{cases}$$

for all $e \in E$.

The next theorem shows that the switching operation preserves valid inequalities and facets of the cut polytope.

4.1.5 Theorem. *Let $w \in \mathbb{R}^E, w_0 \in \mathbb{R}$ and $S \subseteq V$. The following are equivalent:*

(i) The inequality $w^\top x \le w_0$ is valid (resp. facet inducing) for $\mathrm{CUT}^{01}(G)$.

(ii) The inequality $(w^{\delta_G(S)})^\top x \le w_0 - w(\delta_G(S))$ is valid (resp. facet inducing) for $\mathrm{CUT}^{01}(G)$.

Similarly, the following are equivalent:

(i) The inequality $w^\top x \le w_0$ is valid (resp. facet inducing) for $\mathrm{CUT}^{\pm 1}(G)$.

(ii) The inequality $(w^{\delta_G(S)})^\top x \le w_0$ is valid (resp. facet inducing) for $\mathrm{CUT}^{\pm 1}(G)$.

For a proof of this result see [44, Section 26.3]. A pair of inequalities as given in Theorem 4.1.5 are called *switching equivalent*.

4.2 The metric polytope

We have already seen that linear optimization over the cut polytope models the maximum cut problem, well known to be NP-hard [93]. This justifies the need for obtaining tractable relaxations of the cut polytope. In this section we introduce one of the most extensively studied polyhedral relaxations of the cut polytope.

4.2.1 Definition. *The* metric polytope *of a graph $G = (V, E)$, denoted by* $\mathrm{MET}^{01}(G)$, *is the polytope defined by the following linear inequalities:*

$$0 \leq x_e \leq 1 \quad \forall e \in E, \tag{4.2}$$

$$x(F) - x(C \setminus F) \leq |F| - 1, \tag{4.3}$$

for all circuits C of G and for all odd cardinality subsets $F \subseteq C$.

Inequalities of the form (4.3) are called *circuit inequalities* and notice that the well known triangle inequalities are special instances of circuit inequalities.

We introduced the metric polytope as a tractable relaxation for the cut polytope. However, this is not apparent from (4.3), since the number of defining inequalities of the metric polytope is exponential in number of nodes of the graph. Nevertheless, it is known that the separation problem for the metric polytope can be solved in polynomial time [19] (see also [44, Section 27.3.1]). Using the fundamental results of Grötschel, Lovász and Schrijver this implies that linear optimization over the metric polytope can be done in polynomial time [58].

Alternatively, the fact that we can optimize efficiently over the metric polytope of a graph can be seen since $\mathrm{MET}^{01}(G)$ can be expressed as the projection of a polytope living a higher dimension which has a compact description. This polytope is known as the *semimetric polytope* and is equal to $\mathrm{MET}^{01}(K_n)$. The semimetric polytope lies in $\binom{n}{2}$-dimensional space and is defined by the following $4\binom{n}{3}$ inequalities

$$x_{ij} - x_{ik} - x_{jk} \leq 0 \quad \text{and} \quad x_{ij} + x_{ik} + x_{jk} \leq 2,$$

for all distinct $i, j, k \in [n]$. The following theorem shows that optimization over $\mathrm{MET}^{01}(G)$ can be expressed as polynomial size linear program.

4.2.2 Theorem. *[18] For any graph G we have that*

$$\mathrm{MET}^{01}(G) = \pi_E(\mathrm{MET}^{01}(K_n)).$$

For a proof of this fact the reader is referred to [44, Theorem 27.3.3].

In this thesis we will usually work with the metric polytope in ± 1 variables, denoted by $\mathrm{MET}^{\pm 1}(G)$, in which case its linear inequality description is given by:

$$-1 \leq x_e \leq 1 \quad \forall e \in E, \tag{4.4}$$

$$x(C \setminus F) - x(F) \leq |C| - 2, \tag{4.5}$$

for all circuits C of G and for all odd cardinality subsets $F \subseteq C$.

Next we make the simple observation that the metric polytope is a relaxation of the cut polytope, i.e., that for any graph G we have the inclusion

$$\mathrm{CUT}^{01}(G) \subseteq \mathrm{MET}^{01}(G). \tag{4.6}$$

For this we need to show that all the cut vectors $\delta_G(S)$ belong to $\mathrm{MET}^{01}(G)$. Let $S \subseteq V$ and C a circuit in G. Then $\delta_G(S)(C) = |\delta_G(S) \cap C|$ is an even number which implies that $\delta_G(S)$ satisfies (4.3). The next theorem characterizes the graphs for which (4.6) holds with equality.

4.2.3 Theorem. *[17] For any graph G we have that*

$$\mathrm{CUT}(G) = \mathrm{MET}(G) \text{ if and only if } G \text{ has no } K_5\text{-minor}.$$

It will be useful in later sections to identify which of the defining inequalities of the metric polytope are facet inducing.

4.2.4 Theorem. *[19] For a graph $G = (V, E)$ we have that*

(i) Inequality (4.2) *defines a facet of* $\mathrm{MET}^{01}(G)$ *if and only if e does not belong to any triangle of G.*

(ii) Inequality (4.3) *defines a facet of* $\mathrm{MET}^{01}(G)$ *if and only if C is a chordless circuit.*

We continue with an observation that will be useful in Chapter 9.

4.2.5 Lemma. *For a fixed chrodless circuit C all the circuit inequalities given in* (4.5) *are switching equivalent.*

Proof. Say $C = C_n$, where n is odd. One of the circuit inequalities given in (4.5) is

$$-x(C_n) \leq |C| - 2. \tag{4.7}$$

We will show that any other circuit inequality is switching equivalent to (4.7). Let $F \subseteq C_n$ with $|F|$ odd, and consider the corresponding circuit inequality $x(C \setminus F) - x(F) \leq |C| - 2$. Since $|C|$ and $|F|$ are both odd it follows that $|C \setminus F|$ is even and thus $C \setminus F$ is a cut of C_n. Thus, we can change the signs along $C \setminus F$ to get (4.7). Similarly, for even n, every circuit inequality is switching equivalent to

$$x(C \setminus e) - x_e \leq |C| - 2, \tag{4.8}$$

and thus the claim follows. □

The last result in this section shows that if G is obtained as the clique k-sum ($k \leq 3$) of two graphs G_1 and G_2, the inequality description of $\mathrm{CUT}(G)$ can be obtained by combining the inequality descriptions of $\mathrm{CUT}(G_1)$ and $\mathrm{CUT}(G_2)$.

4.2.6 Theorem. *[17] Consider a graph G obtained as the clique k-sum ($k \leq 3$) of graphs G_1 and G_2. Then a linear inequality description of* $\mathrm{CUT}(G)$ *is obtained by juxtaposing the linear inequality descriptions of* $\mathrm{CUT}(G_1)$ *and* $\mathrm{CUT}(G_2)$ *and identifying the variables corresponding to edges contained in $V_1 \cap V_2$.*

4.3 The elliptope of a graph

In this section we introduce the elliptope of a graph, one of the most extensively studied non-polyhedral relaxations for the cut polytope. This non-polyhedral relaxation is relevant for optimization purposes as one can optimize a linear function over the elliptope in polynomial time using semidefinite programing.

Given a graph $G = (V = [n], E)$, π_E denotes the projection from $\mathscr{S}^n$ onto the subspace $\mathbb{R}^E$ indexed by the edge set of G, i.e.,

$$\pi_E : \mathscr{S}^n \mapsto \mathbb{R}^E \qquad X \mapsto (X_{ij})_{ij \in E}. \tag{4.9}$$

4.3.1 Definition. *The* elliptope *of a graph G is defined as*

$$\mathscr{E}(G) = \pi_E(\mathscr{E}_n).$$

The study of the elliptope is motivated by its relevance to the positive semidefinite matrix completion problem. Clearly, the elements of $\mathscr{E}(G)$ can be seen as the G-partial symmetric matrices that admit a completion to a full correlation matrix. Consequently, deciding whether a G-partial matrix admits a psd completion is equivalent to deciding membership in $\mathscr{E}(G)$.

A necessary condition for a G-partial matrix to admit a psd completion is that every completely specified minor should also be positive semidefinite. This condition is also sufficient for chordal graphs as illustrated in the next theorem.

4.3.2 Theorem. *[55] For any graph G we have that*

$$\mathscr{E}(G) \subseteq \{x \in [-1,1]^{|E|} \;:\; x_K \in \mathscr{E}(K) \;\textit{for all cliques K in G}\},$$

and equality holds if and only if G is chordal.

Here x_K denotes the restriction of the vector $x \in \mathbb{R}^E$ to those entries which are indexed by edges in K.

Any matrix $X = (x_{ij}) \in \mathscr{E}_n$ has its diagonal entries all equal to 1. Hence, all its entries lie in $[-1,1]$ and thus they can be parametrized as $x_{ij} = \cos(\pi a_{ij})$ where $a_{ij} \in [0,1]$. This parametrization allows us to state conditions for the membership of a G-partial matrix in $\mathscr{E}(G)$ in terms of linear inequalities in the $a'_{ij}s$. A first result in this direction characterizes the elliptope of a circuit. Throughout this section, for a vector $x \in \mathbb{R}^E$ we set $\cos x = (\cos x_{ij}) \in \mathbb{R}^E$.

4.3.3 Theorem. *[20] For a circuit C we have that*

$$\mathscr{E}(C) = \{\cos \pi a : a \in \mathrm{MET}^{01}(C)\}.$$

To gain some intuition concerning Theorem 4.3.3, notice that for $C = K_3$ we recover the well-known result that the 3-by-3 matrix

$$\begin{pmatrix} 1 & \cos a & \cos\gamma \\ \cos a & 1 & \cos\beta \\ \cos\gamma & \cos\beta & 1 \end{pmatrix}$$

is positive semidefinite if and only if

$$a \le \beta + \gamma,\ \beta \le a + \gamma,\ \gamma \le a + \beta,\ \text{and}\ a + \beta + \gamma \le 2\pi.$$

One can relate the elliptope of a graph with the metric polytope as follows

4.3.4 Theorem. *[74] For any graph G we have that*

$$\mathscr{E}(G) \subseteq \{\cos \pi a : a \in \mathrm{MET}^{01}(G)\},$$

with equality if and only if G does not have a K_4-minor.

The next result shows that the elliptope of a graph is indeed a relaxation of the cut polytope.

4.3.5 Theorem. *[74] For any graph G we have that*

$$\mathrm{CUT}^{\pm 1}(G) \subseteq \mathscr{E}(G),$$

with equality if and only if G does not have a K_3-minor.

We continue with a simple lemma that will be useful in Chapter 9.

4.3.6 Lemma. [74, Corollary 4.7] *For p even, we have $ce \in \mathscr{E}(C_p)$ for all $c \in [-1,1]$. For p odd, we have $ce \in \mathscr{E}(C_p)$ if and only if $-\cos\frac{\pi}{p} \leq c \leq 1$.*

Proof. Let $c \in [-1,1]$ and consider $a \in [0,1]$ such that $c = \cos \pi a$. By Theorem 4.3.4 we have that $\mathscr{E}(C_p) = \cos(\pi \mathrm{MET}^{01}(C_p))$ which implies that $ce^\top \in \mathscr{E}(C_p)$ if and only if $ae^\top \in \mathrm{MET}^{01}(C_p)$. By the definition of $\mathrm{MET}^{01}(C_p)$, the latter condition is equivalent to $a(2|F| - |C_p|) \leq |F| - 1$ for all $F \subseteq C_p$ with $|F|$ odd. As $a \geq 0$ this condition is trivially satisfied for all $F \subseteq C$ with $2|F| - |C_p| \leq 0$. This implies that $ae^\top \in \mathscr{E}(C_p)$ if and only if $a \leq \min\{(|F|-1)/(2|F|-|C_p|) : 2|F| - |C_p| > 0,\ F \subseteq C_p,\ |F| \text{ odd }\}$.

Consider first the case where p is even. Then $(|F|-1)/(2|F|-|C_p|) \geq 1$ for every $F \subseteq C_p$ with $2|F| - |C_p| > 0$ and $|F|$ odd which implies that $ae^\top \in \mathrm{MET}^{01}(C_p)$ for all $a \in [0,1]$. Next let us consider the case when p is odd. Setting $F = C_p$ we obtain that $a \leq (p-1)/p$. For $F \subseteq C_p$ with $|F| \leq p-2$ it is easy to check that $(p-1)/p \leq (|F|-1)/(2|F|-p)$ and thus the claim follows. □

We conclude this section by collecting some geometric properties of the elliptope of a graph that will be useful for Chapter 10. Since the affine image of a compact set is compact we immediately get the following:

4.3.7 Lemma. *The elliptope of a graph G is a compact and convex subset of $\mathbb{R}^{|E|}$.*

The next lemma is useful in the study of the extreme points of elliptope $\mathscr{E}(G)$.

4.3.8 Lemma. *Let $x \in \mathscr{E}(G)$, let $X \in \mathscr{E}_n$ be a rank r completion of x with Gram representation $\{u_1, \ldots, u_n\}$ in $\mathbb{R}^r$ and let U be the $r \times n$ matrix with columns $u_1, \ldots, u_n$. Set*

$$U_{ij} = \frac{u_i u_j^\top + u_j u_i^\top}{2},\ \mathscr{U}_V = \langle U_{ii} : i \in V\rangle,\ \mathscr{U}_E = \langle U_{ij} : \{i,j\} \in E\rangle \subseteq \mathscr{S}^r. \qquad (4.10)$$

If x is an extreme point of $\mathscr{E}(G)$ then $\mathscr{U}_E \subseteq \mathscr{U}_V$.

Proof. Assume for contradiction that $\mathscr{U}_E \not\subseteq \mathscr{U}_V$. Then there exists a matrix $R \in \mathscr{U}_V^\perp \setminus \mathscr{U}_E^\perp$. Set $Z = U^\top R U = (\langle R, U_{ij}\rangle)_{i,j=1}^n \in \mathscr{S}^n$ and notice that since $R \in \mathscr{U}_V^\perp$, the matrix Z is a perturbation of X (recall (3.8) and (4.10)). By the definition of perturbation this means that there exists some $\epsilon > 0$ such that $X \pm \epsilon Z \in \mathscr{E}_n$. As $R \notin \mathscr{U}_E^\perp$, it follows that $Z_{ij} = \langle R, U_{ij}\rangle \neq 0$ for some edge $\{i,j\} \in E$ and thus the vectors $\pi_E(X+\epsilon Z)$ and $\pi_E(X-\epsilon Z)$ are both distinct from x. Then the equality $x = \pi_E(X+\epsilon Z)/2 + \pi_E(X-\epsilon Z)/2$, leads to a contradiction since by assumption $x \in \operatorname{ext}\mathscr{E}(G)$. □

Given a vector $x \in \mathscr{E}(G)$, its *fiber*, denoted by $\mathrm{fib}(x)$, is the set of all psd completions of x in $\mathscr{E}_n$, i.e,

$$\mathrm{fib}(x) = \{X \in \mathscr{E}_n : \pi_E(X) = x\}.$$

We conclude with a lemma that will be used in Chapter 10.

4.3.9 Lemma. *For a vector $x \in \mathscr{E}(G)$ we have that*

(i) $x \in \mathrm{ext}\,\mathscr{E}(G)$ *if and only if* $\mathrm{fib}(x)$ *is a face of* $\mathscr{E}_n$.

(ii) If $x \in \mathrm{ext}\,\mathscr{E}(G)$ *then* $\mathrm{ext}\,\mathrm{fib}(x) \subseteq \mathrm{ext}\,\mathscr{E}_n$.

Proof. (i) Say $x \in \mathrm{ext}\,\mathscr{E}(G)$ and let $\lambda A + (1-\lambda)B \in \mathrm{fib}(x)$, where $A, B \in \mathscr{E}_n$ and $\lambda \in (0,1)$. Then $x = \lambda \pi_E(A) + (1-\lambda)\pi_E(B) \in \mathscr{E}(G)$ and since $x \in \mathrm{ext}\,\mathscr{E}(G)$ this implies that $A, B \in \mathrm{fib}(x)$. The other direction is similar.

(ii) The assumption combined with (i) imply that $\mathrm{fib}(x)$ is a face of $\mathscr{E}_n$ and the claim follows from Lemma 2.1.5. $\square$

5

The Gram dimension of a graph

In this chapter we introduce a new graph parameter, denoted by $\mathrm{gd}(\cdot)$, which we call the Gram dimension of a graph. It is defined as the smallest integer $k \geq 1$ such that any partial real symmetric matrix, whose entries are specified on the diagonal and at the off-diagonal positions corresponding to edges of G, can be completed to a positive semidefinite matrix of rank at most k (assuming a positive semidefinite completion exists). We show that for any fixed integer $k \geq 1$ the class of graphs satisfying $\mathrm{gd}(G) \leq k$ is minor closed and hence, by the graph minor theorem, it can be characterized by a finite list of forbidden minors. For $k \leq 3$ the only minimal forbidden minor is K_{k+1}. Our main result in this chapter is to identify the forbidden minors for the case $k = 4$.

The content of this chapter is based on joint work with M. Laurent [83, 82].

5.1 Introduction

The problem of completing a partial matrix to a full positive semidefinite (psd) matrix is one of the most extensively studied matrix completion problems. A particular instance of this problem is the completion problem for correlation matrices arising in probability and statistics, and it is also closely related to the completion problem for Euclidean distance matrices with applications, e.g., to sensor network localization and to molecular conformation in chemistry.

Among all psd completions of a partial matrix, the ones with the lowest possible rank are of particular importance. Indeed, the rank of a matrix is often a good measure of the complexity of the data it represents. As an example, it is well known that the minimum dimension of a Euclidean embedding of a finite metric space can be expressed as the rank of an appropriate psd matrix (see e.g. [44]). Moreover, in applications, one is often interested in embeddings in low dimension, say 2 or 3.

In this chapter we focus on the question of existence of low rank psd completions. Our approach is combinatorial, so we look for conditions on the graph specified entries permitting to guarantee the existence of low rank completions. This is captured by the notion of *Gram dimension* of a graph which we introduce below.

We start by recalling some basic definitions. Given a simple and undirected graph G on n nodes, a *G-partial matrix* is a real symmetric n-by-n matrix whose entries are specified on the diagonal and at the off-diagonal positions corresponding to the edges of G. A *G-partial psd matrix* is a G-partial matrix with the additional property that every fully specified principal submatrix is psd. A G-partial psd matrix that admits at least one completion to a full psd matrix is called *completable*. Throughout this chapter we denote the set of G-partial matrices that are completable as $\mathcal{S}_+(G)$ and the set of G-partial matrices that admit a positive definite completion as $\mathcal{S}_{++}(G)$.

5.1.1 Definition. *The Gram dimension of a graph $G = ([n], E)$, denoted by* gd(G)*, is defined as the smallest integer $k \geq 1$ such that, for any matrix $X \in \mathcal{S}_+^n$, there exists another matrix $X' \in \mathcal{S}_+^n$ with rank at most k satisfying*

$$X_{ii} = X'_{ii} \textit{ for all } i \in [n] \textit{ and } X_{ij} = X'_{ij} \textit{ for all } ij \in E.$$

Equivalently, the Gram dimension of a graph is equal to the smallest integer $k \geq 1$ such that every G-partial psd matrix which is completable also has a psd completion of rank at most k.

Notice that the Gram dimension is well defined and satisfies $\mathrm{gd}(G) \leq |V(G)|$ since in the definition one can always take $X' = X$.

As a warm-up example we observe that $\mathrm{gd}(K_n) = n$; the upper bound is clear as $|V(K_n)| = n$ and the lower bound follows by considering $X = I_n$, i.e., the identity matrix of size n-by-n.

Yet another equivalent way of rephrasing the notion of Gram dimension is in terms of ranks of feasible solutions to certain semidefinite programs. Indeed, the Gram dimension of a graph $G = (V = [n], E)$ is at most k if and only if the set

$$S(G, a) = \{X \succeq 0 : X_{ii} = a_{ii} \ \forall i \in [n] \text{ and } X_{ij} = a_{ij} \ \forall ij \in E\}$$

contains a matrix of rank at most k for all $a \in \mathbb{R}^{V \cup E}$ for which $S(G, a)$ is not empty. The set $S(G, a)$ is a typical instance of a spectrahedron; recall Section 3.2.

Specializing the general bound from Corollary 3.2.2 to the spectrahedron $S(G, a)$, we obtain the bound

$$\mathrm{gd}(G) \leq \left\lfloor \frac{\sqrt{1 + 8(|V| + |E|)} - 1}{2} \right\rfloor.$$

For the complete graph $G = K_n$ this bound is equal to n and since $\mathrm{gd}(K_n) = n$ it is tight. As we will see in this chapter one can get other bounds depending on the structure of G; for instance, $\mathrm{gd}(G)$ is at most the treewidth of G plus 1 (cf. Theorem 5.2.8).

As we will see in Section 5.2, for any fixed integer $k \geq 1$ the class of graphs with $\mathrm{gd}(G) \leq k$ is closed under taking minors, hence it can be characterized by a finite list of minimal forbidden minors. Our main result in this chapter is such a characterization for each integer $k \leq 4$.

Main Theorem. *For $k \leq 3$, $\mathrm{gd}(G) \leq k$ if and only if G has no K_{k+1} minor. For $k = 4$, $\mathrm{gd}(G) \leq 4$ if and only if G has no K_5 and $K_{2,2,2}$ minors.*

5.2 Definitions and basic properties

In this section we give some useful reformulations of the parameter $\mathrm{gd}(\cdot)$ and we prove some basic properties that will be useful for later sections.

5.2.1 Definition. *Given a graph $G = (V,E)$ and a vector $a \in \mathbb{R}^{V \cup E}$, a Gram representation of a in $\mathbb{R}^k$ consists of a set of vectors $p_1, \dots, p_n \in \mathbb{R}^k$ such that*

$$p_i^{\mathsf{T}} p_j = a_{ij} \ \forall ij \in V \cup E.$$

The Gram dimension of a vector $a \in \mathscr{S}_+(G)$, denoted as $\mathrm{gd}(G,a)$, is the smallest integer $k \geq 1$ for which a has a Gram representation in $\mathbb{R}^k$.

Then, a vector $a \in \mathscr{S}_+(G)$ with $\mathrm{gd}(G,a) \leq k$ corresponds to a G-partial matrix that has at least one psd completion of rank at most k.

Using the fact the every psd matrix is the Gram matrix of some family of vectors, the graph parameter $\mathrm{gd}(\cdot)$ can be reformulated as follows:

5.2.2 Definition. *The Gram dimension of a graph $G = (V,E)$ is defined as*

$$\mathrm{gd}(G) = \max_{a \in \mathscr{S}_+(G)} \mathrm{gd}(G,a). \tag{5.1}$$

5.2.3 Remark. *In this thesis we are primarily concerned with vectors in $\mathbb{R}^{V \cup E}$ that admit a Gram representation by unit vectors. Any such vector has the form (e,a) where $a \in \mathbb{R}^E$ and e is the all-ones vector of size $|V|$. For ease of notation, for any vector $a \in \mathbb{R}^E$, we write $\mathrm{gd}(G,a)$ in place of $\mathrm{gd}(G,(e,a))$ and this is a convention we follow throughout this thesis. Then, for a vector $a \in \mathbb{R}^E$, $\mathrm{gd}(G,a)$ is equal to the smallest $k \geq 1$ for which a has a Gram representation by unit vectors in $\mathbb{R}^k$.*

We now observe that the maximization in (5.1) can be restricted to all vectors $a \in \mathscr{E}(G)$ (where all diagonal entries are implicitly taken to be equal to 1).

5.2.4 Lemma. *For any graph G we have that*

$$\mathrm{gd}(G) = \max_{a \in \mathscr{E}(G)} \mathrm{gd}(G,a). \tag{5.2}$$

Proof. As a first step we show that the maximization in (5.1) can be restricted to vectors $a \in \mathscr{S}_+(G)$ satisfying $a_{ii} \neq 0$ for all $i \in [n]$. Indeed, for a vector $a \in \mathscr{S}_+(G)$ with $a_{ii} = 0$ for some i, define the new vector $\tilde{a}$ which coincides with a everywhere except at the (i,i)-th entry where $\tilde{a}_{ii} = 1$. Then $\tilde{a} \in \mathscr{S}_+(G)$ and $\mathrm{gd}(G,a) \leq \mathrm{gd}(G,\tilde{a})$. Iterating, this implies the claim. Lastly, given a vector $a \in \mathscr{S}_+(G)$ with $a_{ii} \neq 0$ for all $i \in [n]$, we can scale it and define a new vector $\tilde{a} \in \mathscr{E}(G)$ by setting $\tilde{a}_{ij} = a_{ij}/\sqrt{a_{ii}a_{jj}}$ for all $ij \in V \cup E$. It is straightforward to check that $\mathrm{gd}(G,a) = \mathrm{gd}(G,\tilde{a})$ and thus the lemma follows. □

In Chapter 10 we will study another related graph parameter, called the extreme Gram dimension of a graph, defined as follows:

$$\mathrm{egd}(G) = \max_{a \in \mathrm{ext}\,\mathscr{E}(G)} \mathrm{gd}(G,a).$$

That is, $\mathrm{egd}(G)$ is the maximum Gram dimension over the extreme points of $\mathscr{E}(G)$.

Next we investigate the behavior of the graph parameter $\mathrm{gd}(\cdot)$ under the operation of edge deletion and edge contraction.

5.2.5 Lemma. *The graph parameter* $\mathrm{gd}(\cdot)$ *is monotone nonincreasing with respect to edge deletion and contraction:* $\mathrm{gd}(G\backslash e), \mathrm{gd}(G/e) \le \mathrm{gd}(G)$ *for any edge* $e \in E$.

Proof. Let $G = ([n], E)$ and $e \in E$. It follows directly from the definition that $\mathrm{gd}(G\backslash e) \le \mathrm{gd}(G)$. It remains to show that $\mathrm{gd}(G/e) \le \mathrm{gd}(G)$. Say e is the edge $(n-1, n)$, $G/e = ([n-1], E')$, and set $k = \mathrm{gd}(G)$. Consider vectors $p_1, \dots, p_{n-1}$ labeling the nodes of G/e. As $\mathrm{gd}(G) = k$, there exists a family of vectors $q_1, \dots, q_n \in \mathbb{R}^k$ such that

$$q_i^\mathsf{T} q_j = p_i^\mathsf{T} p_j \text{ for all } ij \in [n] \cup E, \tag{5.3}$$

where we define $p_n = p_{n-1}$. Notice that, by applying (5.3) to the pairs ij with $i, j \in \{n-1, n\}$, we get that $q_{n-1} = q_n$. We now show that $p_i^\mathsf{T} p_j = q_i^\mathsf{T} q_j$ for all $ij \in [n-1] \cup E'$ which will imply the claim. Recall that the edge set E' consists of the edges of G not containing node n and of the edges $(i, n-1)$ for all $(i, n) \in E$. If $ij \in E' \cap E$ then we are done by (5.3). Lastly, if $(i, n-1) \in E'$ where $(i, n) \in E$, then (5.3) implies that $q_i^\mathsf{T} q_n = p_i^\mathsf{T} p_n$, and thus, as $q_{n-1} = q_n$ and $p_{n-1} = p_n$, the claim follows. □

Our next goal is to investigate the behavior of $\mathrm{gd}(\cdot)$ with respect to the clique sum operation.

5.2.6 Lemma. *If G is the clique sum of two graphs G_1 and G_2, then*

$$\mathrm{gd}(G) = \max\{\mathrm{gd}(G_1), \mathrm{gd}(G_2)\}.$$

Proof. The claim follows directly from Lemma 2.3.11. □

As a first application of Lemmas 5.2.5 and 5.2.6, we obtain that the Gram dimension is monotone nonincreasing under node deletion.

5.2.7 Lemma. *For any node u of G, we have that* $\mathrm{gd}(G \setminus u) \le \mathrm{gd}(G)$*; moreover, equality holds if u is an isolated node.*

Proof. After deleting all edges adjacent to u in G, we obtain a graph (call it H) which can be seen as the clique 0-sum of node u and the graph $G \setminus u$. Thus, $\mathrm{gd}(H) = \mathrm{gd}(G \setminus u)$ (by Lemma 5.2.6) and $\mathrm{gd}(H) \le \mathrm{gd}(G)$ (by Lemma 5.2.5). □

As another application we can bound the Gram dimension of a graph in terms of its treewidth.

5.2.8 Theorem. *For any graph G,* $\mathrm{gd}(G) \le \mathrm{tw}(G) + 1$.

Proof. Setting $\mathrm{tw}(G) = k$, by the definition of treewidth we have that G is a subgraph of a clique sum of complete graphs on $k+1$ nodes. Combining Lemma 5.2.6 with the fact that $\mathrm{gd}(K_{k+1}) = k+1$ the claim follows. □

Next, we relate the Gram dimension of a graph G and of its suspension ∇G. Given a vector $x \in \mathbb{R}^{V \cup E}$, extend it to a vector $y \in \mathbb{R}^{V(\nabla G) \cup E(\nabla G)}$ by setting $y_{ij} = x_{ij}$ for all $ij \in V \cup E$, $y_{0i} = 0$ for all $i \in [n]$ and letting $y_{00} > 0$ be an arbitrary positive scalar. Then, if $x \in \mathscr{S}_+(G)$, we have that $y \in \mathscr{S}_+(\nabla G)$ and moreover $\mathrm{gd}(\nabla G, y) = \mathrm{gd}(G, x) + 1$. This shows that $\mathrm{gd}(\nabla G) \ge \mathrm{gd}(G) + 1$.

The next lemma shows that this holds with equality.

5.2.9 Lemma. *For any graph G,* $\mathrm{gd}(\nabla G) = \mathrm{gd}(G) + 1$.

Proof. Set $k = \mathrm{gd}(G)$; we show that $\mathrm{gd}(\nabla G) \le k+1$. For this, let $X \in \mathcal{S}_+^{n+1}$, written in block-form as $X = \begin{pmatrix} \alpha & a^T \\ a & A \end{pmatrix}$, where $A \in \mathcal{S}_+^n$ and the first row/column is indexed by the apex node 0 of ∇G. If $\alpha = 0$ then $a = 0$, $\pi_{VE}(A)$ has a Gram representation in $\mathbb{R}^k$ and thus $\pi_{V(\nabla G)E(\nabla G)}(X)$ too. Assume now $\alpha > 0$ and without loss of generality $\alpha = 1$. Consider the Schur complement Y of X with respect to the entry $\alpha = 1$, given by $Y = A - aa^T$ (recall Theorem 2.3.8). As $Y \in \mathcal{S}_+^n$, there exists $Z \in \mathcal{S}_+^n$ such that $\mathrm{rank}(Z) \le k$ and $\pi_{VE}(Z) = \pi_{VE}(Y)$. Define the matrix

$$X' := \begin{pmatrix} 1 & a^T \\ a & aa^T \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}.$$

Then, $\mathrm{rank}(X') = \mathrm{rank}(Z) + 1 \le k+1$. Moreover, X' and X coincide at all diagonal entries as well as at all entries corresponding to edges of ∇G. This concludes the proof that $\mathrm{gd}(\nabla G) \le k+1$. □

Throughout this chapter we denote by $\mathcal{G}_k$ the class of graphs G for which $\mathrm{gd}(G) \le k$. In view of Lemmas 5.2.5 and 5.2.7, $\mathcal{G}_k$ is closed under taking minors. Hence, by the celebrated graph minor theorem of [115], it can be characterized by finitely many minimal forbidden minors.

It is easy to see that for all integers $n \ge 1$ the graph K_n is a minimal forbidden minor for $\mathcal{G}_{n-1}$. The fact that it is forbidden follows since $\mathrm{gd}(K_n) = n$ so it remains to show minimality. Indeed, contracting any edge of the graph K_n gives the graph K_{n-1} which has Gram dimension $n-1$. On the other hand, deleting any edge of K_n we get a graph which is the clique sum of two copies of K_{n-1}, in which case we are done by Theorem 5.2.8.

In the next theorem we determine the full list of minimal forbidden minors for the class $\mathcal{G}_k$ when $k \le 3$. For the case $k = 1$ we have that $\mathrm{gd}(G) = 1$ if and only if G does not contain any edge. The only interesting cases are for $k \in \{2, 3\}$.

5.2.10 Theorem. *For $k \le 3$, $\mathrm{gd}(G) \le k$ if and only if G has no minor K_{k+1}.*

Proof. For any graph G we have the following chain of implications

$$\mathrm{gd}(G) \le 2 \Longrightarrow K_3 \not\preceq G \Longrightarrow \mathrm{tw}(G) \le 1 \Longrightarrow \mathrm{gd}(G) \le 2,$$

where the last implication follows from Theorem 5.2.8 and the second to last implication from Theorem 2.2.2. This gives the characterization of graphs with Gram dimension at most 2. Similarly, for any graph G we have that

$$\mathrm{gd}(G) \le 3 \Longrightarrow K_4 \not\preceq G \Longrightarrow \mathrm{tw}(G) \le 2 \Longrightarrow \mathrm{gd}(G) \le 3,$$

which gives the characterization of graphs with Gram dimension at most 3. □

As an application, Theorem 5.2.10 implies that for the circuit graph we have $\mathrm{gd}(C_n) \le 3$. In the following lemma, we derive a characterization of the partial matrices $a \in \mathcal{E}(C_n)$ admitting a Gram realization in $\mathbb{R}^2$. This result will be generalized to arbitrary graphs in Chapter 7.

5.2.11 Lemma. *Consider the vector $a = (\cos\vartheta_1, \cos\vartheta_2, \ldots, \cos\vartheta_n) \in \mathcal{E}(C_n)$, where $\vartheta_1, \ldots, \vartheta_n \in [0, \pi]$. Then $\mathrm{gd}(C_n, a) \le 2$ if and only if there exist $\epsilon \in \{\pm 1\}^n$ and $k \in \mathbb{Z}$ such that $\sum_{i=1}^n \epsilon_i \vartheta_i = 2k\pi$.*

Proof. Assume that $\mathrm{gd}(C_n, a) \leq 2$ and let $u_1, \ldots, u_n \in \mathbb{R}^2$ be unit vectors such that $u_i^\top u_{i+1} = \cos\vartheta_i$ for all $i \in [n]$ (setting $u_{n+1} = u_1$). We may assume that $u_1 = (1,0)^\top$. Then, the condition $u_1^\top u_2 = \cos\vartheta_1$ implies that the angle between u_1 and u_2 is equal to $\cos\vartheta_1$ and thus $u_2 = (\cos(\epsilon_1\vartheta_1), \sin(\epsilon_1\vartheta_1))^\top$ for some $\epsilon_1 \in \{\pm 1\}$. Analogously, $u_2^\top u_3 = \cos\vartheta_2$ implies that $u_3 = (\cos(\epsilon_1\vartheta_1 + \epsilon_2\vartheta_2), \sin(\epsilon_1\vartheta_1 + \epsilon_2\vartheta_2))^\top$ for some $\epsilon_2 \in \{\pm 1\}$. Iterating, we find there exists $\epsilon \in \{\pm 1\}^n$ such that $u_i = (\cos(\sum_{j=1}^{i-1} \epsilon_i\vartheta_i), \sin(\sum_{j=1}^{i-1} \epsilon_i\vartheta_i))^\top$ for $i = 1, \ldots, n$. Finally, the condition $u_n^\top u_1 = \cos\vartheta_n = \cos(\sum_{i=1}^{n-1} \epsilon_i\vartheta_i)$ implies $\sum_{i=1}^{n} \epsilon_i\vartheta_i \in 2\pi\mathbb{Z}$. The arguments can be reversed to show the 'if part'. □

5.3 Characterizing graphs with Gram dimension at most four

The next natural question is to characterize the class $\mathcal{G}_4$. As $\mathrm{gd}(K_5) = 5$ and $K_5 \setminus e$ is a clique sum of two copies of K_4 it follows that K_5 is a minimal forbidden minor for $\mathcal{G}_4$. We now show this is also the case for the complete tripartite graph $K_{2,2,2}$.

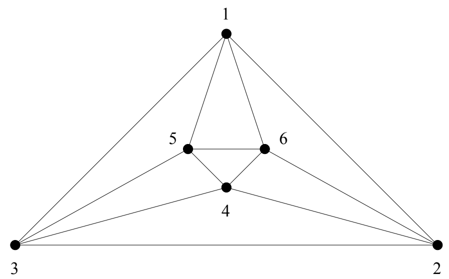

Figure 5.1: The graph $K_{2,2,2}$.

5.3.1 Lemma. *The graph $K_{2,2,2}$ is a minimal forbidden minor for $\mathcal{G}_4$.*

Proof. To show that $\mathrm{gd}(K_{2,2,2}) \geq 5$ we construct a partial matrix $a \in \mathcal{S}_+^n(K_{2,2,2})$ which admits a unique completion to a full positive semidefinite matrix, and this completion has rank 5. For this, number the nodes of $K_{2,2,2}$ as in Figure 5.1. Let e_i ($i \in [5]$) denote the standard unit vectors in $\mathbb{R}^5$. Next, assign vectors p_i ($i \in [6]$) to the nodes of $K_{2,2,2}$, where $p_i = e_i$ for $i \in [5]$ and $p_6 = e_4 + e_5$, and let $a \in \mathcal{S}_+^n(K_{2,2,2})$ be the corresponding partial matrix with the p_i's as Gram representation. By construction the partial matrix a has a psd completion of rank 5 and we now show that this is the unique psd completion of a.

For this, let X be an arbitrary psd completion of a. As the nodes 4,5,6 form a clique in $K_{2,2,2}$, all entries in the principal submatrix $X[4,5,6]$ are specified. Moreover, the chosen Gram representation satisfies the dependency $p_4 + p_5 = p_6$, which gives a linear dependency among the columns of $X[4,5,6]$. Using Lemma 2.3.9, this linear dependency can be extended to a linear dependency among the full columns of X, namely, $X[\cdot,4] + X[\cdot,5] = X[\cdot,6]$. This implies that the three unspecified entries X_{14}, X_{25}, X_{36} are uniquely determined in terms of the specified entries of X.

On the other hand, as $\mathrm{tw}(K_{2,2,2}) \leq 4$, Lemma 5.2.8 implies that $\mathrm{gd}(K_{2,2,2}) \leq 5$ and thus $K_{2,2,2}$ is a forbidden minor for $\mathcal{G}_4$. It remains to show that it is minimal.

Direct case checking shows that deleting or contracting an edge in $K_{2,2,2}$ yields a graph with treewidth at most 3 and thus with Gram dimension at most 4. $\square$

We note in passing that in Chapter 11 we will develop a systematic method for constructing partial matrices with a unique psd completion which will allow us to recover the construction from Lemma 5.3.1.

By Theorem 5.2.8, all graphs with treewidth at most 3 belong to $\mathscr{G}_4$. Moreover, recall that a graph G has $\mathrm{tw}(G) \leq 3$ if and only if G does not have K_5, $K_{2,2,2}$, V_8 and $C_5 \square K_2$ as a minor; cf Theorem 2.2.2. The graphs V_8 and $C_5 \square K_2$ are shown in Figures 5.2 and 5.3, respectively.

These four graphs are natural candidates for being forbidden minors for the class $\mathscr{G}_4$. We have already seen that this is indeed the case for the two graphs K_5 and $K_{2,2,2}$. However, this is not true for V_8 and $C_5 \square K_2$. Both belong to $\mathscr{G}_4$, this will be proved in Section 5.4.5 for V_8 (Theorem 5.4.13) and in Section 5.5 for $C_5 \square K_2$ (Theorem 5.5.1). These two results form the main technical part of this chapter. Using them, we can complete our characterization of the class $\mathscr{G}_4$.

5.3.2 Theorem. *For a graph G, $\mathrm{gd}(G) \leq 4$ if and only if G does not have K_5 or $K_{2,2,2}$ as minors.*

Proof. Necessity follows from Lemmas 5.2.5 and 5.3.1. Sufficiency follows from the following graph theoretical result, obtained by combining Theorem 2.2.2(iii) with Seymour's splitter theorem (for a self-contained proof see [126]): every graph with no K_5 and $K_{2,2,2}$ minors can be obtained as a subgraph of a clique k-sum ($k \leq 2$) of copies of graphs with treewidth at most 3, V_8 and $C_5 \square K_2$. Combining this fact with Theorems 5.4.13, 5.5.1 and Lemmas 5.2.6, 5.2.8 the claim follows. $\square$

5.4 Ingredients of the proof

5.4.1 High level idea

In this section we sketch our approach to show that $\mathrm{gd}(V_8) = \mathrm{gd}(C_5 \square K_2) = 4$.

5.4.1 Definition. *Given a graph $G = (V = [n], E)$, a configuration of G is an assignment of vectors $p_1, \ldots, p_n$ (in some space) to the nodes of G; the pair $(G, \mathbf{p})$ is called a framework. We use the notation $\mathbf{p} = \{p_1, \ldots, p_n\}$ and, for a subset $T \subseteq V$, $\mathbf{p}_T = \{p_i : i \in T\}$. Thus $\mathbf{p} = \mathbf{p}_V$ and we also set $\mathbf{p}_{-i} = \mathbf{p}_{V \setminus \{i\}}$.*

Two configurations $\mathbf{p}, \mathbf{q}$ of G (not necessarily lying in the same space) are said to be equivalent if $p_i^{\mathsf{T}} p_j = q_i^{\mathsf{T}} q_j$ for all $ij \in V \cup E$.

Our objective is to show that the two graphs $G = V_8$, $C_5 \square K_2$ belong to $\mathscr{G}_4$. That is, we must show that, given any $a \in \mathscr{S}_+(G)$, one can construct a Gram representation $\mathbf{q}$ of (G, a) lying in the space $\mathbb{R}^4$.

Along the lines of [25] (which deals with Euclidean distance realizations), our strategy to achieve this is as follows: First, we construct a 'flat' Gram representation $\mathbf{p}$ of (G, a) obtained by maximizing the inner product $p_{i_0}^{\mathsf{T}} p_{j_0}$ along a given pair (i_0, j_0) which is not an edge of G. As suggested in [123] (in the context of Euclidean distance realizations), this configuration $\mathbf{p}$ can be obtained by solving a semidefinite program; then $\mathbf{p}$ corresponds to the Gram representation of an optimal solution X to this program.

In general we cannot yet claim that $\mathbf{p}$ lies in $\mathbb{R}^4$. However, we can derive useful information about $\mathbf{p}$ by using an optimal solution Ω (which will correspond to a 'stress matrix') to the dual semidefinite program. Indeed, the optimality condition $X\Omega = 0$ will imply some linear dependencies among the p_i's that can be used to show the existence of an equivalent representation $\mathbf{q}$ of (G,a) in low dimension. Roughly speaking, most often, these dependencies will force the majority of the p_i's to lie in $\mathbb{R}^4$, and one will be able to rotate each remaining vector p_j about the space spanned by the vectors labeling the neighbors of j into $\mathbb{R}^4$. Showing that the initial representation $\mathbf{p}$ can indeed be 'folded' into $\mathbb{R}^4$ as just described makes up the main body of the proof.

Before going into the details of the proof, we indicate some additional genericity assumptions that can be made without loss of generality on the vector $a \in \mathscr{S}_+(G)$. This will be particularly useful when treating the graph $C_5\Box K_2$.

5.4.2 Genericity assumptions

As observed in Lemma 5.2.4, the Gram dimension $\mathrm{gd}(G)$ is equal to the maximum value of $\mathrm{gd}(G,a)$ taken over all $a \in \mathscr{E}(G)$. We now show using continuity arguments that we can restrict the maximum to be taken over all a lying in a dense subset of $\mathscr{E}(G)$.

5.4.2 Lemma. *Let $\mathscr{D}$ be a dense subset of $\mathscr{E}(G)$. Then*

$$\mathrm{gd}(G) = \max_{a\in\mathscr{D}} \mathrm{gd}(G,a).$$

Proof. Set $k = \max_{a\in\mathscr{D}} \mathrm{gd}(G,a)$ and let $a \in \mathscr{E}(G)$. Since $\mathscr{D}$ is dense in $\mathscr{E}(G)$ there exists a sequence $(d_i)_{i\in\mathbb{N}} \subseteq \mathscr{D}$ converging to a as $i \mapsto \infty$. For every $i \in \mathbb{N}$ there exists a matrix $D_i \in \mathscr{E}_n$ such that $\operatorname{rank} D_i \leq k$ and $d_i = \pi(D_i)$. Since $\mathscr{E}_n$ is compact, the sequence $(D_i)_{i\in\mathbb{N}}$ has a subsequence which converges to $D \in \mathscr{E}_n$. For contradiction, assume that $\operatorname{rank} D > k$. Then, there exists a $(k+1)$-by-$(k+1)$ submatrix of D with nonzero determinant. Since $\lim_{i\in\mathbb{N}} D_i = D$ and $\operatorname{rank} D_i \leq k$ for all $i \in \mathbb{N}$ this gives a contradiction. Thus $\operatorname{rank} D \leq k$ and since $a = \pi(D)$ we get that $\mathrm{gd}(G) \leq \max_{a\in\mathscr{D}} \mathrm{gd}(G,a)$. The converse inequality is always true, so the claim follows. $\square$

For instance, the set $\mathscr{D}$ consisting of all $x \in \mathscr{E}(G)$ that admit a positive definite completion in $\mathscr{E}_n$ is dense in $\mathscr{E}(G)$. We next identify a smaller dense subset $\mathscr{D}^*$ of $\mathscr{D}$ which we will use in our study of $\mathrm{gd}(C_5\Box K_2)$.

5.4.3 Lemma. *Let $\mathscr{D}^*$ be the set of all $a \in \mathscr{E}(G)$ that admit a positive definite completion in $\mathscr{E}_n$ satisfying the following condition: For any circuit C in G, the restriction $a_C = (a_e)_{e\in C}$ of a to C does not admit a Gram representation in $\mathbb{R}^2$. Then the set $\mathscr{D}^*$ is dense in $\mathscr{E}(G)$.*

Proof. We show that $\mathscr{D}^*$ is dense in $\mathscr{D}$. Let $a \in \mathscr{D}$ and set $a = \cos\vartheta$, where $\vartheta \in [0,\pi]^E$. Given a circuit C in G (say of length p), it follows from Lemma 5.2.11 that a_C has a Gram realization in $\mathbb{R}^2$ if and only if $\sum_{i=1}^{p} \epsilon_i\vartheta_i = 2k\pi$ for some $\epsilon \in \{\pm 1\}^p$ and $k \in \mathbb{Z}$ with $|k| \leq p/2$. Let $\mathscr{H}_C$ denote the union of the hyperplanes in $\mathbb{R}^{E(C)}$ defined by these equations. Therefore, $a \notin \mathscr{D}^*$ if and only if $\vartheta \in \cup_C \mathscr{H}_C$, where the union is taken over all circuits C of G. As $\cup_C \mathscr{H}_C$ is a set of measure 0, by perturbing ϑ we can find a sequence $\vartheta^{(i)} \in [0,\pi]^E \setminus \cup_C \mathscr{H}_C$ converging to ϑ as $i \to \infty$. Then the sequence $a^{(i)} := \cos\vartheta^{(i)}$ tends to a as $i \to \infty$ and, for all i large enough, $a^{(i)} \in \mathscr{D}^*$. This shows that $\mathscr{D}^*$ is a dense subset of $\mathscr{D}$ and thus of $\mathscr{E}(G)$. $\square$

Lastly, Lemma 5.4.2 combined with Lemma 5.4.3 imply the following:

5.4.4 Corollary. *For any graph $G = ([n], E)$, $\mathrm{gd}(G) = \max \mathrm{gd}(G, a)$, where the maximum is over all $a \in \mathscr{E}(G)$ admitting a positive definite completion and whose restriction to any circuit of G has no Gram representation in the plane.*

5.4.3 Semidefinite programming formulation

We now describe how to model the 'flattening' procedure using semidefinite programming (sdp) and how to obtain a 'stress matrix' using sdp duality.

Let $G = (V = [n], E)$ be a graph and let $e_0 = (i_0, j_0)$ be a non-edge of G (i.e., $i_0 \neq j_0$ and $e_0 \notin E$). Let $a \in \mathscr{S}_+(G)$ be a partial positive semidefinite matrix for which we want to show the existence of a Gram representation in a small dimensional space. For this consider the semidefinite program:

$$\max \langle E_{i_0 j_0}, X\rangle \text{ s.t. } \langle E_{ij}, X\rangle = a_{ij} \ (ij \in V \cup E), \ \ X \succeq 0, \tag{5.4}$$

where $E_{ij} = (e_i e_j^{\mathsf{T}} + e_j e_i^{\mathsf{T}})/2$ and $e_1, \ldots, e_n$ are the standard unit vectors in $\mathbb{R}^n$. The dual semidefinite program of (5.4) reads:

$$\min \sum_{ij \in V \cup E} w_{ij} a_{ij} \text{ s.t. } \Omega = \sum_{ij \in V \cup E} w_{ij} E_{ij} - E_{i_0 j_0} \succeq 0. \tag{5.5}$$

5.4.5 Theorem. *Consider a graph $G = ([n], E)$, a pair $e_0 = (i_0, j_0) \notin E$, and let $a \in \mathscr{S}_{++}(G)$. Then there exists a Gram realization $\mathbf{p} = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^k$ (for some $k \geq 1$) of (G, a) and a matrix $\Omega = (w_{ij}) \succeq 0$ satisfying*

$$w_{i_0 j_0} \neq 0, \tag{5.6}$$

$$w_{ij} = 0 \ \textit{ for all } ij \notin V \cup E \cup \{e_0\}, \tag{5.7}$$

$$w_{ii} p_i + \sum_{j \,:\, ij \in E \cup \{e_0\}} w_{ij} p_j = 0 \textit{ for all } i \in [n], \tag{5.8}$$

$$\dim\langle p_i, p_j\rangle = 2 \ \textit{ for all } ij \in E. \tag{5.9}$$

We refer to equation (5.8) as the equilibrium condition *at vertex i and we refer to edge (i_0, j_0) as the* stressed edge.

Proof. Consider the semidefinite program (5.4) and its dual program (5.5). By assumption, a has a positive definite completion, hence the program (5.4) is strictly feasible. Choosing $w_{ii} = 2$ for $i \in [n]$ and $w_{ij} = 0$ for $ij \in E$, we see that the dual program (5.5) is also strictly feasible. Hence there is no duality gap and the optimal values are attained in both programs. Let (X, Ω) be a pair of primal-dual optimal solutions. Then (X, Ω) satisfies the optimality condition: $\langle X, \Omega\rangle = 0$ or, equivalently, $X\Omega = 0$. Say X has rank k and let $\mathbf{p} = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^k$ be a Gram representation of X. Now it suffices to observe that the condition $X\Omega = 0$ can be reformulated as the equilibrium conditions (5.8) (recall (2.2)). The conditions (5.6) and (5.7) follow from the form of the dual program (5.5). Lastly, as $a \in \mathscr{S}_{++}(G)$ it follows that the 2-by-2 minor that corresponds to any edge $ij \in E$ has full rank. This shows that (5.9) is valid for any feasible solution of (5.4). □

Note that, using the variant of Farkas' lemma for semidefinite programming given in Lemma 3.1.5 one can show the existence of a nonzero positive semidefinite matrix $\Omega = (w_{ij})$ satisfying (5.7) and the equilibrium conditions (5.8) also in the case when the semidefinite program (5.4) is not strictly feasible, however now with $w_{i_0 j_0} = 0$. This remark will be useful in the exceptional case considered in Section 5.5.5 where we will have to solve again a semidefinite program of the form (5.4); however this program will have additional conditions imposing that some of the p_i's are pinned so that one cannot anymore assume strict feasibility (see the proof of Lemma 5.5.11).

5.4.4 Useful lemmas

We start with some definitions about stressed frameworks and then we establish some basic tools that we will repeatedly use later in our proof for V_8 and $C_5 \square K_2$. For a matrix $\Omega = (w_{ij}) \in \mathscr{S}^n$ its *support graph* is the graph $\mathscr{S}(\Omega)$ with node set $[n]$ and with edges the pairs (i, j) with $w_{ij} \neq 0$.

5.4.6 Definition. (Stressed framework $(H, \mathbf{p}, \Omega)$) *Consider a framework $(H = (V = [n], F), \mathbf{p})$. A nonzero matrix $\Omega = (w_{ij}) \in \mathscr{S}^n$ is called a* stress matrix *for the framework $(H, \mathbf{p})$ if its support graph $\mathscr{S}(\Omega)$ is contained in H (i.e., $w_{ij} = 0$ for all $ij \notin V \cup F$) and Ω satisfies the equilibrium condition*

$$w_{ii} p_i + \sum_{j : ij \in F} w_{ij} p_j = 0 \quad \forall i \in V. \tag{5.10}$$

Then the triple $(H, \mathbf{p}, \Omega)$ is called a stressed framework*, and a* psd stressed framework *if moreover $\Omega \succeq 0$.*

We let V_Ω denote the set of nodes $i \in V$ for which $w_{ij} \neq 0$ for some $j \in V$. A node $i \in V$ is said to be a 0-node *when $w_{ij} = 0$ for all $j \in V$. Hence, $V \setminus V_\Omega$ is the set of all 0-nodes and, when $\Omega \succeq 0$, i is a 0-node if and only if $w_{ii} = 0$.*

The support graph $\mathscr{S}(\Omega)$ of Ω is called the stressed graph*; its edges are called the* stressed edges *of H and the nodes $i \in V_\Omega$ are called the* stressed nodes.

*Given an integer $t \geq 1$, a node $i \in V$ is said to be a t-*node *if its degree in the stressed graph $\mathscr{S}(\Omega)$ is equal to t.*

Throughout we will deal with stressed frameworks $(H, \mathbf{p}, \Omega)$ obtained by applying Theorem 5.4.5. Hence the graph H arises by adding a new edge e_0 to a given graph G, which we then denote as $H = \widehat{G}$, as indicated below.

5.4.7 Definition. (Extended graph $\widehat{G}$) *Given a graph $G = (V = [n], E)$ and a fixed pair $e_0 = i_0 j_0$ not belonging to E, we set $\widehat{G} = (V, \widehat{E} = E \cup \{e_0\})$.*

We now group some useful lemmas which we will use in order to show that a given framework $(H, \mathbf{p})$ admits an equivalent configuration in lower dimension.

The stress matrix provides some linear dependencies among the vectors p_i labeling the stressed nodes, but it gives no information about the vectors labeling the 0-nodes. However, if we have a set S of 0-nodes forming a stable set, then we can use the following lemma in order to 'fold' the corresponding vectors p_i $(i \in S)$ in a lower dimensional space.

5.4.8 Lemma. (Folding a stable set) *Let $(H = (V, F), \mathbf{p})$ be a framework and let $T \subseteq V$. Assume that $S = V \setminus T$ is a stable set in H, that each node $i \in S$ has degree at most $k - 1$ in H, and that $\dim\langle \mathbf{p}_T \rangle \leq k$. Then there exists a configuration $\mathbf{q}$ of H in $\mathbb{R}^k$ which is equivalent to $(H, \mathbf{p})$.*

Proof. Fix a node $i \in S$. Let $N[i]$ denote the closed neighborhood of i in H consisting of i and the nodes adjacent to i. By assumption, $|N[i]| \leq k$ and both sets of vectors $\mathbf{p}_T$ and $\mathbf{p}_{N[i]}$ have rank at most k. Then one can find an orthogonal matrix P mapping all vectors p_j ($j \in T \cup N[i]$) into the space $\mathbb{R}^k$. Indeed, the Gram matrix of the vectors of $\mathbf{p}_T$ and the Gram matrix of the vectors in $\mathbf{p}_{N[i]}$ have both have rank at most k and thus admit a common psd completion of rank k (recall Lemma 2.3.11 and its proof). Repeat this construction with every other node of S. As no two nodes of S are adjacent, this produces a configuration $\mathbf{q}$ in $\mathbb{R}^k$ which is equivalent to $(H, \mathbf{p})$. □

The next lemma uses the stress matrix to upper bound the dimension of a given stressed configuration.

5.4.9 Lemma. (Bounding the dimension) *Let* $(H = (V = [n], F), \mathbf{p}, \Omega)$ *be a psd stressed framework. Then* $\dim\langle \mathbf{p}_V \rangle \leq n - 2$, *except* $\dim\langle \mathbf{p}_V \rangle \leq n - 1$ *if* $\mathscr{S}(\Omega)$ *is a clique.*

Proof. Let X denote the Gram matrix of the p_i's, so that $\text{rank}(X) = \dim\langle \mathbf{p}_V \rangle$. By assumption, $X\Omega = 0$ (and Ω is a nonzero psd matrix) which implies that $\text{rank} X \leq n - 1$. Moreover, if $\mathscr{S}(\Omega)$ is not a clique, then $\text{rank}\,\Omega \geq 2$ and thus $\text{rank} X \leq n - 2$. □

The next lemma indicates how 1-nodes can occur in a stressed framework.

5.4.10 Lemma. *Let* $(H = (V, F), \mathbf{p}, \Omega)$ *be a psd stressed framework. If node* i *is a 1-node in the stressed graph* $\mathscr{S}(\Omega)$, *i.e., there is a unique edge* $ij \in F$ *such that* $w_{ij} \neq 0$, *then* $\dim\langle p_i, p_j \rangle \leq 1$.

Proof. Directly, using the equilibrium condition (5.10) at node i. □

We now consider 2-nodes in a stressed framework. As we will use Schur complements, we recall the definition in the form which we will use here. For a matrix $\Omega = (w_{ij}) \in \mathscr{S}^n$ and $i \in [n]$ with $w_{ii} \neq 0$, the Schur complement of Ω with respect to its (i,i)-entry is the matrix, denoted as $\Omega_{-i} = (w'_{jk})_{j,k \in [n] \setminus \{i\}} \in \mathscr{S}^{n-1}$, with entries $w'_{jk} = w_{jk} - w_{ik} w_{ij} / w_{ii}$ for $j, k \in [n] \setminus \{i\}$. Then, $\Omega \succeq 0$ if and only if $w_{ii} > 0$ and $\Omega_{-i} \succeq 0$. We also need the following notion of 'contracting a degree 2 node' in a graph.

5.4.11 Definition. *Let* $H = (V, F)$ *be a graph, let* $i \in V$ *be a node of degree 2 in* H *which is adjacent to nodes* $i_1, i_2 \in V$. *The graph obtained by* contracting node i *in* H *is the graph* H/i *with node set* $V \setminus \{i\}$ *and with edge set* $F/i = F \setminus \{ii_1, ii_2\} \cup \{i_1 i_2\}$ *(ignoring multiple edges).*

5.4.12 Lemma. (Contracting a 2-node) *Let* $(H = (V, F), \mathbf{p}, \Omega)$ *be a psd stressed framework, let* $i \in V$ *be a 2-node in the stressed graph* $\mathscr{S}(\Omega)$ *and set* $N(i) = \{i_1, i_2\}$. *Then* $p_i \in \langle p_{i_1}, p_{i_2} \rangle$ *and thus* $\dim\langle \mathbf{p} \rangle = \dim\langle \mathbf{p}_{-i} \rangle$.

Moreover, if the stressed graph $\mathscr{S}(\Omega)$ *is not equal to the clique on* $\{i, i_1, i_2\}$, *then* $(H/i, \mathbf{p}_{-i}, \Omega_{-i})$ *is also a psd stressed framework.*

Proof. The equilibrium condition at node i implies $p_i \in \langle p_{i_1}, p_{i_2} \rangle$. Note that the Schur complement Ω_{-i} of Ω with respect to the (i,i)-entry w_{ii} has entries $w'_{i_1 i_2} = w_{i_1 i_2} - w_{ii_1} w_{ii_2} / w_{ii}$, $w'_{i_r i_r} = w_{i_r i_r} - w_{ii_r}^2 / w_{ii}$ for $r = 1, 2$, and $w'_{jk} = w_{jk}$ for all other edges jk of H/i. As $\Omega \succeq 0$ we also have $\Omega_{-i} \succeq 0$. Moreover, $\Omega_{-i} \neq 0$. Indeed, if

$i_1i_2 \notin F$ then $w_{i_1i_2} = 0$ and as $w_{ii_1}, w_{ii_2} \neq 0$ it follows that $w'_{i_1i_2} \neq 0$. On the other hand, if $i_1i_2 \in F$ then, as $\mathscr{S}(\Omega)$ is not the clique on $\{i, i_1, i_2\}$, there is another edge jk of H/i in $\mathscr{S}(\Omega)$ which implies that $w'_{jk} = w_{jk} \neq 0$.

In order to show that Ω_{-i} is a stress matrix for $(H/i, \mathbf{p}_{-i})$, it suffices to check the stress equilibrium at the nodes i_1 and i_2. To fix ideas consider node i_1. Then we can rewrite $w'_{i_1i_1}p_{i_1} + w'_{i_1i_2}p_{i_2} + \sum_{j\in N(i_1)\setminus\{i_2\}} w'_{i_1j}p_j$ as

$$(\sum_j w_{i_1j}p_j) - \left(w_{ii}p_i + w_{ii_1}p_{i_1} + w_{ii_2}p_{i_2}\right) w_{ii_1}/w_{ii},$$

where both terms are equal to 0 using the equilibrium conditions of $(\Omega, \mathbf{p})$ at nodes i_1 and i. □

In our proofs we will apply Lemma 5.4.12 iteratively to contract a set I containing several 2-nodes. Of course, in order to obtain useful information, we want to be able to claim that, after contraction, we obtain a stressed framework $(H/I, \mathbf{p}_{V\setminus I}, \Omega_{-I})$, i.e., with $\Omega_{-I} \neq 0$. Problems might occur if at some step we get a stressed graph which is a clique on 3 nodes. Note that this can happen only when a connected component of the stressed graph is a circuit. However, when we will apply this operation of contracting 2-nodes to the case of $G = C_5\Box K_2$, we will make sure that this situation cannot happen; that is, we will show that we may assume that the stressed graph does not have a connected component which is a circuit (see Remark 5.5.7 in Section 5.5.1).

5.4.5 The graph V_8 has Gram dimension 4

Let $V_8 = (V = [8], E)$ be the graph shown in Figure 5.2. In this section we use the tools developed above to show that V_8 has Gram dimension 4.

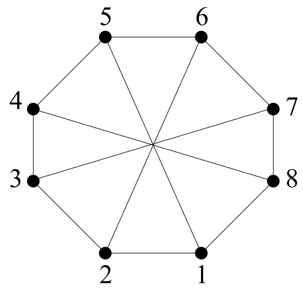

Figure 5.2: The graph V_8.

5.4.13 Theorem. *The graph V_8 has Gram dimension 4.*

Proof. Set $G = V_8 = ([8], E)$. Clearly $\mathrm{gd}(G) \geq 4$ since K_4 is a minor of G. Fix $a \in \mathscr{S}_{++}(G)$; we show that (G, a) has a Gram realization in $\mathbb{R}^4$. For this we first apply Theorem 5.4.5. As stretched edge e_0, we choose the pair $e_0 = (1, 4)$ and we denote by $\widehat{G} = ([8], \widehat{E} = E \cup \{(1, 4)\})$ the extended graph obtained by adding the stretched pair $(1, 4)$ to G. Let $\mathbf{p}$ be the initial Gram realization of (G, a) and let $\Omega = (w_{ij})$ be the corresponding stress matrix obtained by applying Theorem 5.4.5. We now show how to construct from $\mathbf{p}$ an equivalent realization q of (G, a) lying in $\mathbb{R}^4$.

In view of Lemma 5.4.8, we know that we are done if we can find a subset $S \subseteq V$ which is stable in the graph G and satisfies $\dim\langle \mathbf{p}_{V\setminus S}\rangle \leq 4$. This permits to conclude when the stressed graph contains 1-nodes. Indeed suppose that there is a 1-node in the stressed graph $\mathscr{S}(\Omega)$. In view of Lemma 5.4.10 and (5.9), this can only be node 1 (or node 4) (i.e., the end points of the stretched pair) and then we have $\dim\langle p_1, p_4\rangle \leq 1$. Then, choosing the stable set $S = \{2,5,7\}$, we have $\dim\langle \mathbf{p}_{V\setminus S}\rangle \leq 4$ and we can conclude using Lemma 5.4.8. Hence we can now assume that there is no 1-node in the stressed graph $\mathscr{S}(\Omega)$.

Next, observe that we are done in any of the following two cases:

(i) There exists a set $T \subseteq V$ with $|T| = 4$ and $\dim\langle \mathbf{p}_T\rangle \leq 2$.

(ii) There exists a set $T \subseteq V$ of cardinality $|T| = 3$ such that T does not consist of three consecutive nodes on the circuit $(1,2,\ldots,8)$ and $\dim\langle \mathbf{p}_T\rangle \leq 2$.

Indeed, in case (i) (resp., case (ii)), there is a stable set $S \subseteq V \setminus T$ of cardinality $|S| = 2$ (resp., $|S| = 3$), so that $|V \setminus (S \cup T)| = 2$ and thus $\dim\langle \mathbf{p}_{V\setminus S}\rangle \leq \dim\langle \mathbf{p}_T\rangle + \dim\langle \mathbf{p}_{V\setminus(S\cup T)}\rangle \leq 2 + 2 = 4$.

Hence we may assume that we are not in the situation of cases (i) and (ii).

Assume first that one of the nodes in $\{5,6,7,8\}$ is a 0-node. Then all of them are 0-nodes. Indeed, if (say) 5 is a 0-node and 6 is not a 0-node then the equilibrium equation at node 6 implies that $\dim\langle p_6, p_7, p_2\rangle \leq 2$, so that we are in the situation of case (ii). As nodes 1, 4 are not 1-nodes, the stressed graph $\mathscr{S}(\Omega)$ is the circuit $(1,2,3,4)$. Using Lemma 5.4.12, we deduce that $\dim\langle p_1,p_2,p_3,p_4\rangle \leq 2$ and thus we are in the situation of case (i) above.

Assume now that none of the nodes in $\{5,6,7,8\}$ is a 0-node but one of the nodes in $\{2,3\}$ is a 0-node. Then both nodes 2 and 3 are 0-nodes (else we are in the situation of case (ii)). Therefore, both nodes 6 and 7 are 2-nodes. Applying Lemma 5.4.12, after contracting both nodes 6,7, we obtain a stressed framework on $\{1,4,5,8\}$ and thus $\dim\langle \mathbf{p}_{V\setminus\{2,3\}}\rangle = \dim\langle p_1,p_4,p_5,p_8\rangle$. Using Lemma 5.4.9, we deduce that $\dim\langle p_1,p_4,p_5,p_8\rangle \leq 3$. Therefore, $\dim\langle \mathbf{p}_{V\setminus\{3\}}\rangle \leq 4$ and one can find a new realization $\mathbf{q}$ in $\mathbb{R}^4$ equivalent to $(G,\mathbf{p})$ using Lemma 5.4.8.

Finally assume that none of the nodes in $\{2,3,5,6,7,8\}$ is a 0-node. We show that $\langle \mathbf{p}\rangle = \langle p_2,p_3,p_6,p_7\rangle$. Using the equilibrium equation at node 6 we find that $\dim\langle p_2,p_5,p_6,p_7\rangle \leq 3$. Moreover, $\dim\langle p_2,p_6,p_7\rangle = 3$ (else we are in case (ii) above). Hence $p_5 \in \langle p_2,p_6,p_7\rangle$. Analogously, the equilibrium equations at nodes 7,2,3 give that $p_8,p_1,p_4 \in \langle p_2,p_3,p_6,p_7\rangle$, respectively. □

5.5 The graph $C_5\square K_2$ has Gram dimension 4

This section is devoted to proving that the graph $C_5\square K_2$ has Gram dimension 4. The analysis is considerably more involved than the analysis for V_8. Figure 5.3 shows two drawings of $C_5\square K_2$, the second one making its symmetries more apparent.

5.5.1 Theorem. *The graph $C_5\square K_2$ has Gram dimension 4.*

Throughout this section we set $G = C_5\square K_2 = (V = [10], E)$. Clearly, $\mathrm{gd}(G) \geq 4$ because K_4 is a minor of G. In order to show that $\mathrm{gd}(G) \leq 4$, we must show that

$\mathrm{gd}(G,a) \le 4$ for any $a \in \mathcal{S}_{++}(G)$. Moreover, in view of Corollary 5.4.4, it suffices to show this for all $a \in \mathcal{S}_{++}(G)$ satisfying the following 'genericity' property: For any Gram realization $\mathbf{p}$ of (G,a),

$$\dim\langle \mathbf{p}_C \rangle \ge 3 \text{ for any circuit } C \text{ in } G. \tag{5.11}$$

From now on, we fix $a \in \mathcal{S}_{++}(G)$ satisfying this genericity property. Our objective is to show that there exists a Gram realization of (G,a) in $\mathbb{R}^4$.

Again we use Theorem 5.4.5 to construct an initial Gram realization $\mathbf{p}$ of (G,a). As stretched edge e_0, we choose the pair $e_0 = (3,8)$ and we denote by $\widehat{G} = ([8], \widehat{E} = E \cup \{(3,8)\})$ the extended graph obtained by adding the stretched pair $(3,8)$ to G. By Theorem 5.4.5, we also have a stress matrix Ω so that $(\widehat{G}, \mathbf{p}, \Omega)$ is a psd stressed framework. Our objective is now to construct from $\mathbf{p}$ another Gram realization $\mathbf{q}$ of (G,a) lying in $\mathbb{R}^4$.

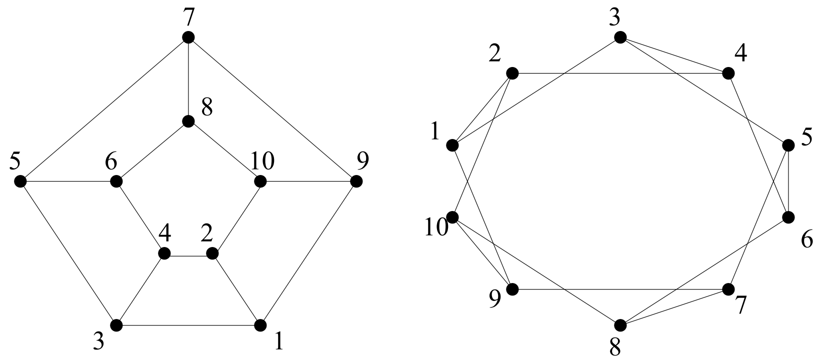

Figure 5.3: Two drawings of the graph $C_5 \square K_2$.

5.5.1 Additional useful lemmas

First we deal with the case when $\dim\langle p_i, p_j\rangle = 1$ for some pair (i,j) of distinct nodes. As $a \in \mathcal{S}_{++}(G)$, this can only happen when $(i,j) \notin E$.

5.5.2 Lemma. *If $\dim\langle p_i, p_j\rangle = 1$ for some pair $(i,j) \notin E$, then there is a configuration in $\mathbb{R}^4$ equivalent to $(G, \mathbf{p})$.*

Proof. By assumption, $p_i = \epsilon p_j$ for some scalar $\epsilon \neq 0$. Up to symmetry there are three cases to consider: (i) $(i,j) = (1,5)$, (ii) $(i,j) = (1,4)$ and (iii) $(i,j) = (1,6)$. Consider first case (i) when $(i,j) = (1,5)$, so $p_1 = \epsilon p_5$. Set $V' = V \setminus \{1\}$. Let $G' = (V', E')$ be the graph on V' obtained from G by deleting node 1 and adding the edges $(2,5)$ and $(5,9)$ (in other words, get G' by identifying nodes 1 and 5 in G). Let X' be the Gram matrix of the vectors p_i $(i \in V')$ and define $a' = (X'_{jk})_{jk \in V' \cup E'} \in \mathcal{S}_{+}(G')$. First we show that (G', a') has a Gram realization in $\mathbb{R}^4$. For this, consider the graph H obtained from G by deleting both nodes 1 and 5. Then G' is a subgraph of ∇H and thus $\mathrm{gd}(G') \le \mathrm{gd}(\nabla H) = \mathrm{gd}(H) + 1$ (recall Lemma 5.2.9). As $\mathrm{tw}(H) \le 2$ it follows that $\mathrm{gd}(H) \le 3$ and thus $\mathrm{gd}(G') \le 4$. Finally, if $\mathbf{q}_{V'}$ is a Gram realization in $\mathbb{R}^4$ of (G', a') then, setting $q_1 = \epsilon q_5$, we obtain a Gram realization $\mathbf{q}$ of (G,a) in $\mathbb{R}^4$.

Cases (ii), (iii) are analogous, using the fact that the graph H obtained from G by deleting nodes 1 and 4, or nodes 1 and 6, respectively, has $\mathrm{tw}(H) \le 2$. $\square$

We now consider the case when the stressed graph might have a circuit as a connected component.

5.5.3 Lemma. *Let C be a circuit in $\widehat{G}$. If C is a connected component of $\mathscr{S}(\Omega)$, then* $\dim\langle \mathbf{p}_C\rangle \leq 2$.

Proof. Directly, using Lemma 5.4.12 combined with Lemma 5.4.9. □

Therefore, in view of the genericity assumption (5.11), if a circuit C is a connected component of the stressed graph, then C cannot be a circuit in G and thus C must contain the stretched pair $e_0 = (3,8)$. The next results are useful to handle this case, treated in Corollary 5.5.6 below.

Recall that a graph G' is obtained from G by a $Y\Delta$ transformation if there is a vertex i of degree 3 in G such that G' is obtained by removing the vertex i and adding an edge between each pair of vertices in the neighborhood of i. We will denote this by $G' = Y\Delta_i G$.

5.5.4 Lemma. *Consider two frameworks $G(\mathbf{p})$ and $G'(\mathbf{p}_{-i})$ (where $G' = Y\Delta_i G$) and let $a = (p_i^{\mathsf{T}} p_j) \in \mathscr{E}(G)$ and $a' = (p_i^{\mathsf{T}} p_j) \in \mathscr{E}(G')$. Then we have that*

$$\mathrm{gd}(G,a) \leq \max\{\mathrm{gd}(G',a'),4\}.$$

Proof. The claim follows immediately after noticing that G is contained n the clique 3-sum of G' and K_4. □

Notice that Lemma 5.5.4 implies that if $\mathrm{gd}(G',a') \leq 4$ then we can conclude that $\mathrm{gd}(G,a) \leq 4$. This observation will be used in the following lemma.

5.5.5 Lemma. *Let $N_2(i)$ be the set of nodes at distance 2 from a given node i in G. If* $\dim\langle \mathbf{p}_{N_2(i)}\rangle \leq 3$*, then there is a configuration equivalent to $(G,\mathbf{p})$ in $\mathbb{R}^4$.*

Proof. Say, $i = 1$ so that $N_2(1) = \{4,5,7,10\}$, cf. Figure 5.4. Consider the set $S = \{2,3,6,9\}$ which is stable in G. Let H denote the graph obtained from G in the following way: For each node $i \in S$, delete i and add the clique on $N(i)$. One can verify that H is contained in the clique 4-sum of the two cliques H_1 and H_2 on the node sets $V_1 = \{1,4,5,7,10\}$ and $V_2 = \{4,5,7,8,10\}$, respectively. By assumption, $\dim\langle \mathbf{p}_{V_1}\rangle \leq 4$ and $\dim\langle \mathbf{p}_{V_2}\rangle \leq 4$. Therefore, one can apply an orthogonal transformation and find vectors $q_i \in \mathbb{R}^4$ ($i \in V_1 \cup V_2$) such that $\mathbf{p}_{V_r}$ and $\mathbf{q}_{V_r}$ have the same Gram matrix, for $r = 1,2$. Finally, as $V_1 \cup V_2 = V \setminus S$ and the set S is stable in G, one can extend to a configuration $\mathbf{q}_V$ equivalent to $\mathbf{p}_V$ by applying Lemma 5.4.8. □

5.5.6 Corollary. *If there is a circuit C in $\widehat{G}$ containing the (stretched) edge* $(3,8)$ *such that* $\dim\langle \mathbf{p}_C\rangle \leq 2$*, then there is a configuration equivalent to $(G,\mathbf{p})$ in $\mathbb{R}^4$.*

Proof. If $|C| \geq 7$, pick $i \in V \setminus C$ and note that $\dim\langle \mathbf{p}_{-i}\rangle \leq 4$. If $|C| = 6$, pick a subset $S \subseteq V \setminus C$ of cardinality 2 that is stable in G, so that $\dim\langle \mathbf{p}_{V\setminus S}\rangle \leq 4$. In both cases we conclude using Lemma 5.4.8. Assume now that $|C| = 4$ or 5. Then, one can check that there exists a node i for which $|C \cap N_2(i)| = 3$. For instance, for $C = (3,8,7,5)$, this holds for node $i = 9$, and for $C = (3,8,10,9,1)$ this holds for node $i = 2$. Then, $|C \cap N_2(i)| = 3$ implies $|N_2(i) \setminus C| = 1$ which, combined with $\dim\langle \mathbf{p}_C\rangle \leq 2$, gives $\dim\langle \mathbf{p}_{N_2(i)}\rangle \leq 3$. Therefore, we are done by Lemma 5.5.5. □

5.5.7 Remark. *Combining Lemma 5.5.2 and the genericity assumption from Corollary 5.4.4 we will assume from now on that*

$$\dim\langle p_i, p_j\rangle = 2 \textit{ for all } i \neq j \in V. \tag{5.12}$$

Hence there is no 1-node in the stressed graph. Moreover, the stressed graph must have at least three nodes. Furthermore, we will assume that no circuit C of $\widehat{G}$ satisfies $\dim\langle \mathbf{p}_C\rangle \leq 2$. *Therefore, the stressed graph does not have a connected component which is a circuit (by (5.11), Lemma 5.5.3 and Corollary 5.5.6). Hence we are guaranteed that after contracting several 2-nodes we do obtain a stressed framework (i.e, with a nonzero stress matrix).*

The next two lemmas settle the case when there are sufficiently many 2-nodes.

5.5.8 Lemma. *If there are at least four 2-nodes in the stressed graph $\mathscr{S}(\Omega)$, then there is a configuration equivalent to $(G, \mathbf{p})$ in $\mathbb{R}^4$.*

Proof. Let I be a set of four 2-nodes in $\mathscr{S}(\Omega)$. Hence, $\mathbf{p}_I \subseteq \langle \mathbf{p}_{V\setminus I}\rangle$ and thus we will be done if we can show that $\dim\langle \mathbf{p}_{V\setminus I}\rangle \leq 4$.

After contracting each of the four 2-nodes of I, we obtain a psd stressed framework $(\widehat{G}/I, \mathbf{p}_{V\setminus I}, \Omega')$. Indeed, we can apply Lemma 5.4.12 and obtain a nonzero psd stress matrix Ω' in the contracted graph (recall Remark 5.5.7). If the support graph of Ω' is not a clique, Lemma 5.4.9 implies that $\dim\langle \mathbf{p}_{V\setminus I}\rangle \leq |V \setminus I| - 2 = 4$ and thus we are done.

Assume now that $\mathscr{S}(\Omega')$ is a clique on $T \subseteq V \setminus I$. Then $\dim\langle \mathbf{p}_T\rangle \leq t-1$, $|V \setminus (I \cup T)| = 6 - t$, and $t = |T| \in \{3,4,5\}$. Indeed one cannot have $t \leq 2$ (recall Remark 5.5.7) and one cannot have $t = 6$ since, after contracting the four 2-nodes, at least 4 edges are lost so that there remain at most $16 - 4 = 12 < 15$ edges. Pick a node $u \in V \setminus (I \cup T)$ and set $S' = V \setminus (I \cup T \cup \{u\})$, so that V is partitioned as $I \cup T \cup \{u\} \cup S'$. As $u \notin T$, u is not adjacent to any node of I in the stressed graph. Therefore, $\langle \mathbf{p}_I\rangle \subseteq \langle \mathbf{p}_{V\setminus I\cup\{u\}}\rangle = \langle \mathbf{p}_{T\cup S'}\rangle$. Moreover, we have $\dim\langle \mathbf{p}_{T\cup S'}\rangle \leq \dim\langle \mathbf{p}_T\rangle + |S'| \leq (t-1) + (5-t) = 4$. Therefore, $\dim\langle \mathbf{p}_{V\setminus\{u\}}\rangle \leq 4$. Now we can apply Lemma 5.4.8 and find an equivalent configuration in $\mathbb{R}^4$. □

5.5.9 Lemma. *If there is at least one 0-node and at least three 2-nodes in the stressed graph $\mathscr{S}(\Omega)$, then there is a configuration equivalent to $(G, \mathbf{p})$ in $\mathbb{R}^4$.*

Proof. For $r = 0, 2$, let V_r denote the set of r-nodes and set $n_r = |V_r|$. By assumption, $n_0 \geq 1$ and we can assume $n_2 = 3$ (else apply Lemma 5.5.8). Set $W = V \setminus (V_0 \cup V_2)$. After contracting the three 2-nodes in the stressed framework $(\widehat{G}, \mathbf{p}, \Omega)$, we get a stressed framework $(H, \mathbf{p}_W, \Omega')$ on $|W| = 7 - n_0$ nodes. Moreover, by Remark 5.5.7, $|W| \geq 3$ and thus $n_0 \leq 4$.

Assume first that $\mathscr{S}(\Omega')$ is not a clique. Then $\dim\langle \mathbf{p}_W\rangle \leq |W| - 2 = 5 - n_0$ by Lemma 5.4.9. Now we can conclude using Lemma 5.4.8 since in each of the cases: $n_0 = 1, 2, 3, 4$, one can find a stable set $S \subseteq V_0$ such that $\dim\langle \mathbf{p}_{W\cup(V_0\setminus S)}\rangle \leq 4$.

Assume now that $\mathscr{S}(\Omega')$ is a clique. Then $\dim\langle \mathbf{p}_W\rangle \leq |W| - 1 = 6 - n_0$ by Lemma 5.4.9. Note first that $n_0 \neq 1, 2$. Indeed, if $n_0 = 1$ then, after deleting the 0-node and contracting the three 2-nodes, we have lost at least $3 + 3 = 6$ edges. Hence there remain at most $16 - 6 = 10$ edges in the stressed graph $\mathscr{S}(\Omega')$, which therefore cannot be a clique on six nodes. If $n_0 = 2$ then, after deleting the two 0-nodes and contracting the three 2-nodes, we have lost at least $5 + 3 = 8$ edges. Hence there remain at most $16 - 8 = 8$ edges in the stressed graph $\mathscr{S}(\Omega')$, which therefore cannot be a clique on five nodes. In each of the remaining two cases $n_0 = 3, 4$, one can find a stable set $S \subseteq V_0$ of cardinality 2 (since G contains no clique of size 3) and thus $\dim\langle \mathbf{p}_{W\cup(V_0\setminus S)}\rangle \leq (6-n_0)+(n_0-2) = 4$ and we conclude using Lemma 5.4.8. $\square$

5.5.2 Main proof

In the proof we distinguish two cases: (i) there exists no 0-node, and (ii) there exists at least one 0-node. These two cases are considered, respectively, in Sections 5.5.3 and 5.5.4. In both cases the tools developed in the preceding section permit us to find an equivalent realization in $\mathbb{R}^4$, except in one exceptional situation, occurring in case (ii). This execptional situation is when nodes 1,2,9 and 10 are 0-nodes and all edges of $\widehat{G} \setminus \{1, 2, 9, 10\}$ are stressed. This situation needs a specific treatment which is presented in Section 5.5.5.

5.5.3 There is no 0-node in the stressed graph

In this section we consider the case when each node is stressed in $\mathscr{S}(\Omega)$, i.e., $w_{ii} \neq 0$ for all $i \in [n]$.

5.5.10 Lemma. *Assume that all vertices are stressed in the stressed graph $\mathscr{S}(\Omega)$ and that there exists a circuit C of length 4 in G such that all edges in the cut $\delta(C)$ are stressed, i.e., $w_{ij} \neq 0$ for all edges $ij \in \widehat{E}$ with $i \in C$ and $j \in V \setminus C$. Then* $\dim\langle \mathbf{p}_V\rangle \leq 4$.

Proof. Up to symmetry, there are three types of circuits C of length 4 in G to consider: (i) C does not meet $\{3, 8\}$, i.e., $C = (1, 2, 10, 9)$; or (ii) C contains one of the two nodes 3,8, say node 8, and it contains a node adjacent to the other one, i.e., node 3, like $C = (5, 6, 8, 7)$; or (iii) C contains one of 3,8 but has no node adjacent to the other one, like $C = (7, 8, 10, 9)$.

Recall from (5.12) that $\dim\langle p_i, p_j\rangle = 2$ for all $i \neq j$. Consider first the case (i), when $C = (1, 2, 10, 9)$. We show that the set $\mathbf{p}_C$ spans $\mathbf{p}_V$. The equilibrium conditions at the nodes 1,2,9,10, combined with the fact that $w_{13}, w_{24}, w_{79}, w_{8,10}$ are all nonzero, imply that $p_3, p_4, p_7, p_8 \in \langle \mathbf{p}_C\rangle$. As 6 is not a 0-node, $w_{6i} \neq 0$ for some $i \in \{4, 8\}$. Then, the equilibrium condition at node i implies that $p_6 \in \langle \mathbf{p}_C\rangle$. Analogously for node 5.

Case (ii) when $C = (5, 6, 8, 7)$ can be treated in analogous manner. Just note that the equilibrium conditions applied to nodes 7,5,6 and 8, respectively, imply that $p_9, p_3, p_4, p_{10} \in \langle \mathbf{p}_C\rangle$.

We now consider case (iii) when $C = (7, 8, 10, 9)$. Then one sees directly that $p_1, p_2, p_5 \in \langle \mathbf{p}_C\rangle$. If $w_{24} \neq 0$, then the equilibrium conditions at nodes 2,3,6 imply that $p_4, p_3, p_6 \in \langle \mathbf{p}_C\rangle$ and thus $\langle \mathbf{p}_C\rangle = \langle \mathbf{p}_V\rangle$. Assume now that $w_{24} = 0$, which implies $w_{34}, w_{46} \neq 0$. If $w_{13} \neq 0$, then the equilibrium conditions at nodes 1,3,4 (in this order) imply that $\mathbf{p}_C$ spans p_3, p_4, p_6 and we are done. Assume now that $w_{24} = w_{13} = 0$, so that 1,2,4 are 2-nodes. If there is one more 2-node then we are done by Lemma 5.5.8. Hence we can now assume that $w_{ij} \neq 0$ whenever $(i, j) \neq (2, 4)$ or $(1, 3)$. Using Lemma 5.4.12, we can contract the three 2-nodes 1,2,4 in the psd stressed framework $(\widehat{G}, \mathbf{p}, \Omega)$ and obtain a new psd stressed framework on $V \setminus \{1, 2, 4\}$ where nodes 9, 10 have again degree 2. Again, by Lemma 5.4.12 we can contract these two nodes and get another psd stressed framework on $V \setminus \{1, 2, 4, 9, 10\}$. Finally, using Lemma 5.4.9 we get that $\dim\langle \mathbf{p}_V\rangle = \dim\langle \mathbf{p}_{V\setminus\{1,2,4,9,10\}}\rangle \leq 4$. $\square$

In view of Lemma 5.5.10, we can now assume that, for each circuit C of length 4 in G, there is at least one edge $ij \in \delta(C)$ which is not stressed, i.e., $w_{ij} = 0$. It suffices now to show that this implies the existence of at least four 2-nodes, as we can then conclude using Lemma 5.5.8. For this let us enumerate the cuts $\delta(C)$ of the 4-circuits C in G:

- For $C = (1,2,10,9)$, $\delta(C) = \{(1,3),(2,4),(7,9),(8,10)\}$.
- For $C = (7,9,10,8)$, $\delta(C) = \{(1,9),(2,10),(5,7),(6,8)\}$.
- For $C = (5,6,8,7)$, $\delta(C) = \{(7,9),(8,10),(3,5),(4,6)\}$.
- For $C = (3,5,6,4)$, $\delta(C) = \{(1,3),(2,4),(5,7),(6,8)\}$.
- For $C = (1,3,4,2)$, $\delta(C) = \{(3,5),(4,6),(1,9),(2,10)\}$.

For instance, $w_{24} = 0$ implies that both 2 and 4 are 2-nodes, while $w_{13} = 0$ implies that 1 is a 2-node. One can check that, in order to ensure that each cut $\delta(C)$ contains an edge with zero stress, however without creating a 1-node, at least three edges must have a zero stress and each node of $V \setminus \{3,8\}$ is adjacent to at most two of them. One can then verify that this implies that there are at least four 2-nodes in $\mathscr{S}(\Omega)$. (This can be done by direct case checking).

5.5.4 There is at least one 0-node in the stressed graph

Note that the mapping $\sigma : V \to V$ that permutes the elements in each of the pairs $\{1,10\}$, $\{4,7\}$, $\{5,6\}$, $\{2,9\}$ and $\{3,8\}$ is an automorphism of G. This can be easily seen using the second drawing of $C_5 \square K_2$ in Figure 5.3. Hence, as nodes 3 and 8 are not 0-nodes, up to symmetry, it suffices to consider the following three cases:

- Node 1 is a 0-node.
- Nodes 1, 10 are not 0-nodes and node 4 is a 0-node.
- Nodes 1, 10, 4, 7 are not 0-nodes and one of 5 or 2 is a 0-node.

Node 1 is a 0-node.

It will be useful to use the drawing of $\widehat{G}$ from Figure 5.4. There, the thick edge (3,8) is known to be stressed, the dotted edges are known to be non-stressed (i.e., $w_{ij} = 0$), while the other edges could be stressed or not. In view of Lemma 5.5.9, we can assume that there are at most two 2-nodes (else we are done).

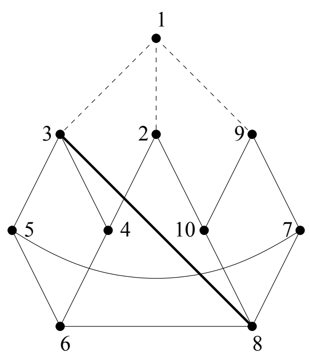

Figure 5.4: A drawing of $\widehat{C_5 \square K_2}$ with 1 as the root node.

Assume first that both nodes 2 and 9 are 0-nodes. Then node 10 too is a 0-

node and each of nodes 4 and 7 is a 0- or 2-node. If both 4,7 are 2-nodes, then all edges in the graph $G\backslash\{1,2,9,10\}$ are stressed. Hence we are in the *exceptional case,* which we will consider in Section 5.5.5 below. If 4 is a 0-node and 7 is a 2-node, then 3,7 must be the only 2-nodes and thus 6 is a 0-node. Hence, the stressed graph is the circuit $C=(3,8,5,7)$, which implies $\dim\langle \mathbf{p}_C\rangle \leq 2$ and thus we can conclude using Corollary 5.5.6. If 4 is a 2-node and 7 is a 0-node, then we find at least two more 2-nodes. Finally, if both 4,7 are 0-nodes, then the stressed graph is the circuit $C=(3,8,6,5)$ and thus we can again conclude using Corollary 5.5.6.

We can now assume that at least one of the two nodes 2,9 is a 2-node. Then, node 3 has degree 3 in the stressed graph. (Indeed, if 3 is a 2-node, then 10 must be a 0-node (else we have three 2-nodes), which implies that 2,9 are 0-nodes, a contradiction.) If exactly one of nodes 2,9 is stressed, one can easily see that there should be at least three 2-nodes. Finally consider the case when both nodes 2,9 are stressed. Then they are the only 2-nodes which implies that all edges of $G\backslash 1$ are stressed. Set $I=\{4,5,8\}$. We show that $\mathbf{p}_I$ spans $\mathbf{p}_{V\backslash\{1\}}$, so that $\mathbf{p}_{\{1,4,5,8\}}$ spans $\mathbf{p}_V$. Indeed, the equilibrium conditions at 3 and 6 imply that $p_3,p_6\in\langle \mathbf{p}_I\rangle$. Next, the equilibrium conditions at 4,5,2,9 imply, respectively, that $p_2\in\langle p_3,p_4,p_6\rangle\subseteq\langle \mathbf{p}_I\rangle$, $p_7\in\langle p_3,p_5,p_6\rangle\subseteq\langle \mathbf{p}_I\rangle$, $p_{10}\in\langle p_2,p_4\rangle\subseteq\langle \mathbf{p}_I\rangle$, and $p_9\in\langle p_7,p_{10}\rangle\subseteq\langle \mathbf{p}_I\rangle$. This concludes the proof.

Nodes 1, 10 are not 0-nodes and node 4 is a 0-node.

It will be useful to use the drawing of $\widehat{G}$ from Figure 5.5. We can assume that node 2 is a 2-node and that node 3 has degree 3 in the stressed graph, since otherwise one would find at least three 2-nodes. Consider first the case when 6 is a 2-node.

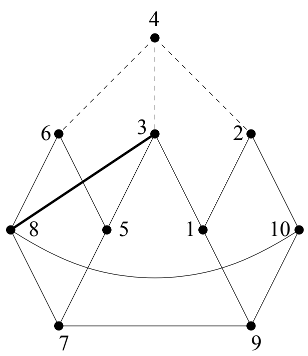

Figure 5.5: A drawing of $\widehat{C_5\square K_2}$ with 4 as the root node.

Then nodes 2 and 6 are the only 2-nodes which implies that all edges in the graph $G\backslash 4$ are stressed. Set $I=\{3,5,7,10\}$. We show that $\mathbf{p}_I$ spans $\mathbf{p}_{V\backslash\{4\}}$, and then we can conclude using Lemma 5.4.8. Indeed, the equilibrium conditions applied, respectively, to nodes 5,6,3,1,2 imply that $p_6\in\langle \mathbf{p}_I\rangle$, $p_8\in\langle p_5,p_6\rangle\subseteq\langle \mathbf{p}_I\rangle$, $p_1\in\langle p_3,p_5,p_8\rangle\subseteq\langle \mathbf{p}_I\rangle$, $p_9\in\langle p_1,p_7,p_{10}\rangle\subseteq\langle \mathbf{p}_I\rangle$, $p_2\in\langle p_1,p_{10}\rangle\subseteq\langle \mathbf{p}_I\rangle$.

Consider now the case when 6 is a 0-node. Then 2 and 5 are the only 2-nodes so that all edges in the graph $G\backslash\{4,6\}$ are stressed. Set $I=\{3,7,10\}$. We show that $\mathbf{p}_I$ spans $\mathbf{p}_{V\backslash\{4,6\}}$, and then we can again conclude using Lemma 5.4.8. Indeed the equilibrium conditions applied, respectively, at nodes 5,8,3,2,1 imply that $p_5,p_8\in$

$\langle \mathbf{p}_I \rangle$, $p_1 \in \langle p_3, p_5, p_8 \rangle \subseteq \langle \mathbf{p}_I \rangle$, $p_2 \in \langle p_1, p_{10} \rangle \subseteq \langle \mathbf{p}_I \rangle$, $p_9 \in \langle p_2, p_8, p_{10} \rangle \subseteq \langle \mathbf{p}_I \rangle$.

Nodes 1, 4, 7, 10 are not 0-nodes and node 5 or 2 is a 0-node.

First, we consider the case when node 5 is a 0-node, so that node 7 is a 2-node. It will be useful to use the drawing of $\widehat{G}$ from Figure 5.6. Recall that by Lemma 5.5.9 we can assume that there exist at most two 2-nodes.

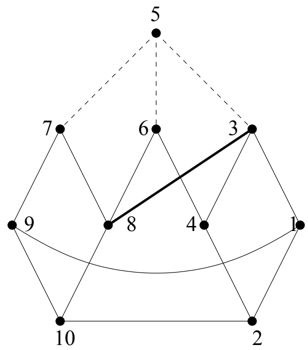

Figure 5.6: A drawing of $\widehat{C_5 \square K_2}$ with 5 as the root node.

If node 6 is a 2-node, then 6 and 7 are the only 2-nodes and thus all edges of the graph $G \backslash 5$ are stressed. Setting $I = \{1, 2, 4, 8\}$, one can verify that $\mathbf{p}_I$ spans $\mathbf{p}_{V \backslash \{5\}}$ and then one can conclude using Lemma 5.4.8.

If node 6 is a 0-node, then nodes 4 and 7 are the only 2-nodes and thus all edges in the graph $G \backslash \{5, 6\}$ are stressed. Setting $I = \{2, 3, 9\}$, one can verify that $\mathbf{p}_I$ spans $\mathbf{p}_{V \backslash \{5,6\}}$. Thus $\mathbf{p}_{\{2,3,9,6\}}$ spans $\mathbf{p}_{V \backslash \{5\}}$ and one can again conclude using Lemma 5.4.8.

Lastly, consider the case when node 2 is a 0-node and nodes 1,4,7,10, 5 are not 0-nodes. Then necessarily node 6 is not a 0-node, for otherwise node 4 would have to be a 0-node. As node 2 is adjacent to nodes 1, 4 and 10 in G, we find three 2-nodes and thus we are done by Lemma 5.5.9.

5.5.5 The exceptional case

In this section we consider the following case which was left open in the first case considered in Section 5.5.4: Nodes 1, 2, 9 and 10 are 0-nodes and all edges of the graph $\widehat{G} \backslash \{1, 2, 9, 10\}$ are stressed.

Then, nodes 4 and 7 are 2-nodes in the stressed graph. After contracting both nodes 4,7, we obtain a stressed graph which is the complete graph on 4 nodes. Hence, using Lemma 5.4.9, we can conclude that $\dim\langle \mathbf{p}_{V_1} \rangle \leq 3$, where $V_1 = V \setminus \{1, 2, 9, 10\}$. Among the nodes 1, 2, 9 and 10, we can find a stable set of size 2. Hence, if $\dim\langle \mathbf{p}_{V_1} \rangle \leq 2$ then, using Lemma 5.4.8, we can find an equivalent configuration in dimension $2 + 2 = 4$ and we are done. From now on we assume that

$$\dim\langle \mathbf{p}_{V_1} \rangle = 3. \tag{5.13}$$

In this case it is not clear how to fold $\mathbf{p}$ in $\mathbb{R}^4$. In order to settle this case, we proceed as in Belk [25]: We fix (or *pin*) the vectors p_i labeling the nodes $i \in V_1$ and we search for another set of vectors p'_k labeling the nodes $k \in V_2 = V \setminus V_1 =$

$\{1,2,9,10\}$, so that $\mathbf{p}_{V_1}\cup\mathbf{p}'_{V_2}$ can be folded into $\mathbb{R}^4$. Again, our starting point is to get such new vectors p'_k ($k\in V_2$) which, together with $\mathbf{p}_{V_1}$, provide a Gram realization of (G,a), by stretching along a second pair e'; namely we stretch the pair $e'=(4,9)\in V_1\times V_2$. As in So and Ye [123], this configuration $\mathbf{p}'_{V_2}$ is again obtained by solving a semidefinite program; details are given below.

Computing $\mathbf{p}'_{V_2}$ via semidefinite programming.

Let $E[V_2]$ denote the set of edges of G contained in V_2 and let $E[V_1,V_2]$ denote the set of edges $(i,k)\in E$ with $i\in V_1$, $k\in V_2$. Moreover, set $|V_1|=n_1\geq|V_2|=n_2$, so the configuration $\mathbf{p}_{V_1}$ lies in $\mathbb{R}^{n_1}$. (Here $n_1=6$, $n_2=4$). We now search for a new configuration $\mathbf{p}'_{V_2}$ by stretching along the pair $(4,9)$. For this we use the following semidefinite program:

$$\max\ \langle F_{49},Z\rangle \text{ such that } \begin{array}{ll} \langle F_{ik},Z\rangle=a_{ik} & \forall ik\in E[V_1,V_2]\\ \langle E_{kl},Z\rangle=a_{kl} & \forall kl\in V_2\cup E[V_2]\\ \langle E_{ij},Z\rangle=0 & \forall i<j,i,j\in V_1\\ \langle E_{ii},Z\rangle=1 & \forall i\in V_1\\ Z\succeq 0. & \end{array} \tag{5.14}$$

Here $a=(p_i^{\mathsf T}p_j)$ for all $i,j\in V\cup E$. Moreover, $E_{ij}\in\mathscr{S}^{n_1+n_2}$ is the symmetric matrix whose ij-th and ji-th entries are $1/2$ and equal to 0 otherwise. Lastly, for $ik\in E[V_1,V_2]$, we define the block matrix $F_{ik}\in\mathscr{S}^{n_1+n_2}$ whose block structure w.r.t. the partition $V=V_1\cup V_2$ is as follows:

$$F_{ik}=\begin{pmatrix}0 & p_ie_k^{\mathsf T}/2\\ e_kp_i^{\mathsf T}/2 & 0\end{pmatrix},$$

where e_k ($k\in[n_2]$) are the standard unit vectors in $\mathbb{R}^{n_2}$; thus all columns of the $V_1\times V_2$-submatrix of F_{ik} are zero, except its k-th column which is equal to $p_i/2$.

Consider a matrix Z feasible for (5.14). Then Z can be written in block form:

$$Z=\begin{pmatrix}I_{n_1} & Y\\ Y^{\mathsf T} & X\end{pmatrix},\ \text{ where } Y\in\mathbb{R}^{n_1\times n_2},\ X\in\mathscr{S}_+^{n_2}. \tag{5.15}$$

Let $y_k\in\mathbb{R}^{n_1}$ ($k\in V_2$) denote the columns of Y. Using Schur complements, we have that $Z\succeq 0$ is equivalent to $X-Y^{\mathsf T}Y\succeq 0$. Say, $X-Y^{\mathsf T}Y$ is the Gram matrix of the vectors $z_k\in\mathbb{R}^{n_2}$ ($k\in V_2$) and define the vectors:

$$p'_i=\begin{pmatrix}p_i\\0\end{pmatrix}\ \text{ for } i\in V_1,\ \text{and } p'_k=\begin{pmatrix}y_k\\z_k\end{pmatrix}\ \text{ for } k\in V_2. \tag{5.16}$$

Then the matrix X is the Gram matrix of the vectors p'_k ($k\in V_2$). Next, we verify that these vectors p'_i ($i\in V_1\cup V_2$) give a Gram realization of (G,a). Indeed, for $ik\in E[V_1,V_2]$, we have that $a_{ik}=\langle F_{ik},Z\rangle=p_i^{\mathsf T}y_k=(p_i,0)^{\mathsf T}(y_k,z_k)=(p'_i)^{\mathsf T}p'_k$. Moreover, for $k,l\in V_2$, we have that $a_{kl}=\langle E_{kl},Z\rangle=X_{kl}=(p'_k)^{\mathsf T}p'_l$.

We now consider the dual semidefinite program of (5.14) which, as we see in Lemma 5.5.11 below, will give us some equilibrium conditions on the new vectors

p'_k ($k \in V_2$). The dual program involves scalar variables w'_{ik} (for $ik \in E[V_1, V_2] \cup V_2 \cup E[V_2]$) and a matrix variable $U' = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}$, and it reads:

$$\begin{aligned} \min \quad & \langle I_{n_1}, U\rangle + \sum_{ik \in E[V_1,V_2]} w'_{ik} a_{ik} + \sum_{kl \in V_2 \cup E[V_2]} w'_{kl} a_{kl} \\ \text{such that} \quad & \Omega' = -F_{49} + U' + \sum_{ik \in E[V_1,V_2]} w'_{ik} F_{ik} + \sum_{kl \in V_2 \cup E[V_2]} w'_{kl} E_{kl} \succeq 0. \end{aligned} \tag{5.17}$$

Notice that the primal program (5.14) attains its maximum since its feasible region is closed and bounded.

5.5.11 Lemma. *Let Z be an optimal solution for* (5.14). *Assume that Z has the block form (5.15), let y_k ($k \in V_2$) denote the columns of Y, let $z_k \in \mathbb{R}^{n_2}$ ($k \in V_2$) be a Gram representation of $X - YY^T$, and let $p'_i \in \mathbb{R}^{n_1+n_2}$ ($i \in V_1 \cup V_2$) be as defined in* (5.16). *Then, there exists a nonzero matrix $\Omega' = (w'_{ij}) \in \mathscr{S}_+^{n_1+n_2}$ satisfying the following conditions:*

$$\begin{aligned} & w'_{ik} = 0 \ \ \forall (i,k) \in (V_1 \times V_2) \setminus (E[V_1,V_2] \cup \{(4,9)\}), \\ & w'_{kl} = 0 \ \ \forall k \neq l \in V_2, kl \notin E[V_2], \end{aligned} \tag{5.18}$$

$$w'_{kk} p'_k + \sum_{i \in V_1 : ki \in E \cup \{(4,9)\}} w'_{ki} \frac{p'_i}{2} + \sum_{l \in V_2 : kl \in E} w'_{kl} p'_l = 0 \ \ \forall k \in V_2, \tag{5.19}$$

$$w'_{kl} \neq 0 \text{ for some } kl \in V_2 \cup E[V_2]. \tag{5.20}$$

Proof. If the primal program (5.14) is strictly feasible, then the dual program (5.17) has an optimal solution Ω'; then Ω' satisfies (5.18), $w'_{49} = -1$ and thus $\Omega' \neq 0$. On the other hand, if (5.14) is not strictly feasible, then Farkas' lemma (Lemma 3.1.5) guarantees the existence of a nonzero matrix $\Omega' \succeq 0$ satisfying (5.18) (now with $w'_{49} = 0$).

Next, we show that in both cases the matrix Ω' satisfies (5.19). Using (5.16), we can split (5.19) into the following two equations:

$$w'_{kk} y_k + \sum_{i \in V_1 : ki \in E \cup \{(4,9)\}} w'_{ki} \frac{p_i}{2} + \sum_{l \in V_2 : kl \in E} w'_{kl} y_l = 0 \ \ \forall k \in V_2, \tag{5.21}$$

$$w'_{kk} z_k + \sum_{l \in V_2 : kl \in E} w'_{kl} z_l = 0 \ \ \forall k \in V_2. \tag{5.22}$$

The key observation is that in both cases Ω' satisfies $Z\Omega' = 0$. Then, writing the matrices Z and Ω' in block form

$$Z = \begin{pmatrix} I_{n_1} & Y \\ Y^{\mathsf{T}} & X \end{pmatrix}, \ \ \Omega' = \begin{pmatrix} \Omega'_1 & \Omega'_{12} \\ (\Omega'_{12})^{\mathsf{T}} & \Omega'_2 \end{pmatrix},$$

the equation $Z\Omega' = 0$ implies that $Y^{\mathsf{T}}\Omega'_{12} + X\Omega'_2 = 0$ and $\Omega'_{12} + Y\Omega'_2 = 0$. Combining these two equations we arrive at the condition $(X - Y^{\mathsf{T}}Y)\Omega'_2 = 0$ which is equivalent to (5.22), since the matrix $X - Y^{\mathsf{T}}Y$ is the Gram matrix of the vectors z_k ($k \in V_2$).

To show that (5.21) holds, substitute $\Omega'_{12} = \sum_{ik \in E[V_1,V_2] \cup \{(4,9)\}} w'_{ik} p'_i e_k^{\mathsf{T}}/2$ in the equation $\Omega'_{12} + Y\Omega'_2 = 0$. Then, for every $k \in V_2$, the condition that the k-th column of the matrix $\Omega'_{12} + Y\Omega'_2$ is equal to zero gives the condition (5.21) at node k.

Lastly, it remains to verify that (5.20) holds, i.e., $\Omega'_2 \neq 0$. Assume for contradiction that $\Omega'_2 = 0$. As Ω' is psd it follows that $\Omega'_{12} = 0$. Then, the condition $Z\Omega' = 0$ gives $0 = \langle Z, \Omega' \rangle = \langle I_{n_1}, \Omega'_1 \rangle$, which implies that $\Omega'_1 = 0$ and thus $\Omega' = 0$, a contradiction. $\square$

Folding $\mathbf{p}'$ into $\mathbb{R}^4$.

We now use the above configuration $\mathbf{p}'$ and the equilibrium conditions (5.19) at the nodes of V_2 to construct a Gram realization of (G, a) in $\mathbb{R}^4$. Recall from (5.16) that $p'_i = (p_i, 0)$ for $i \in V_1$. By (5.12), we may assume that no node $k \in V_2$ is a 1-node with respect to the new stress Ω'. Let us point out again that Lemma 5.5.11 does not guarantee that $w'_{49} \neq 0$ (as opposed to relation (5.6) in Theorem 5.4.5).

By assumption nodes 1,2,9 and 10 are 0-nodes and all other edges of the graph $\widehat{G} \setminus \{1, 2, 9, 10\}$ are stressed w.r.t. the old stress matrix Ω. We begin with the following easy observation about $\mathbf{p}'_{V_1}$.

5.5.12 Lemma. *We have that* $\dim\langle p'_4, p'_7, p'_8 \rangle = \dim\langle p'_3, p'_4, p'_8 \rangle = 3$.

Proof. Using the fact that 1,2,9 and 10 are 0-nodes combined with the equilibrium conditions w.r.t. the old stress matrix Ω it follows that each of these sets spans $\mathbf{p}'_{V_1}$. Lastly, we have that $\dim\langle \mathbf{p}'_{V_1} \rangle = \dim\langle \mathbf{p}_{V_1} \rangle = 3$ by (5.13). $\square$

As an immediate corollary we may assume that

$$p'_k \notin \langle \mathbf{p}'_{V_1} \rangle \ \forall k \in V_2 \tag{5.23}$$

Indeed, if there exists $k \in V_2$ satisfying $p'_k \in \langle \mathbf{p}'_{V_1} \rangle$ then we can find a stable set of size two in $V_2 \setminus \{i\}$ and using Lemma 5.4.8 we can construct an equivalent configuration in $\mathbb{R}^4$. Moreover, we can assume that at most two nodes in V_2 are 0-nodes in $\mathscr{S}(\Omega')$ since, by construction, for the new stress matrix Ω' there exists $kl \in V_2 \cup E[V_2]$ such that $w'_{kl} \neq 0$. This observation motivates the case analysis below. Figure 5.7 below shows the graph containing the relevant support for V_2.

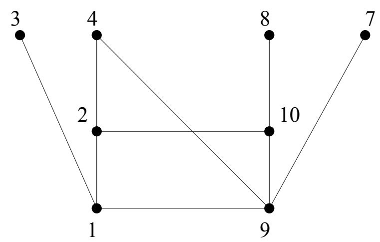

Figure 5.7: The graph containing the relevant support for V_2.

There are exactly two 0-nodes in V_2.

The cases when either 2,9, or 1,10, are 0-nodes are excluded (since then one would have a 1-node). If nodes 1 and 9 are 0-nodes, then the equilibrium conditions at nodes 2 and 10 imply that $p'_4, p'_8 \in \langle p'_2, p'_{10} \rangle$. Since $\dim\langle p'_4, p'_8 \rangle = 2$ by Lemma 5.5.12, we obtain that $\langle p'_2, p'_{10} \rangle = \langle p'_4, p'_8 \rangle \subseteq \langle \mathbf{p}'_{V_1} \rangle$, contradicting (5.23). The case when nodes 9,10 are 0-nodes is similar.

Finally assume that nodes 1,2 are 0-nodes (the case when 2,10 are 0-nodes is analogous). As $w'_{8,10} \neq 0$, the equilibrium condition at node 10 implies that

$p'_8 \in \langle p'_9, p'_{10} \rangle$. If $w'_{49} = 0$ then the equilibrium condition at node 9 implies that $p'_7 \in \langle p'_9, p'_{10} \rangle$. Hence $\langle p'_7, p'_8 \rangle \subseteq \langle p'_9, p'_{10} \rangle$, thus equality holds, contradicting (5.23). If $w'_{49} \neq 0$, then $p'_4 \in \langle p'_7, p'_9, p'_{10} \rangle$ and thus $\langle p'_4, p'_7, p'_8 \rangle \subseteq \langle p'_7, p'_9, p'_{10} \rangle$. Hence equality holds (by Lemma 5.5.12), contradicting again (5.23).

There is exactly one 0-node in V_2.

Suppose first that 9 is the only 0-node in V_2. The equilibrium conditions at nodes 1 and 10 imply that $p'_1 \in \langle p'_3, p'_2 \rangle$ and $p'_{10} \in \langle p'_2, p'_8 \rangle$. Hence $\langle p'_1, p'_{10} \rangle \subseteq \langle \mathbf{p}'_{V_1}, p'_2 \rangle$ and thus $\dim \langle \mathbf{p}'_{V \setminus \{9\}} \rangle = 4$. Then we can conclude using Lemma 5.4.8, as node 9 has degree 3 in the original graph.

Suppose now that node 1 is the only 0-node (the cases when 2 or 10 is the only 0-node are analogous). The equilibrium conditions at nodes 2 and 9 imply that $p'_2 \in \langle p'_4, p'_{10} \rangle$ and $p'_9 \in \langle p'_4, p'_7, p'_{10} \rangle$. Hence, $\langle p'_2, p'_9 \rangle \subseteq \langle \mathbf{p}'_{V_1}, p'_{10} \rangle$ and we can conclude using Lemma 5.4.8.

There is no 0-node in V_2.

We can assume $w'_{ik} \neq 0$ for some $(i,k) \in V_1 \times V_2$ for otherwise we get the stressed circuit $C = (1, 2, 10, 9)$, thus with $\dim \langle \mathbf{p}'_C \rangle = 2$, contradicting our assumption (5.11). We show that $\dim \langle \mathbf{p}'_V \rangle = 4$. For this we discuss according to how many parameters are equal to zero among $w'_{13}, w'_{24}, w'_{8,10}$.

If none is zero, then the equilibrium conditions at nodes 1,2 and 10 imply that $p'_3, p'_4, p'_8 \in \langle \mathbf{p}'_{V_2} \rangle$ and thus Lemma 5.5.12 implies that $\dim(\langle \mathbf{p}'_{V_1} \rangle \cap \langle \mathbf{p}'_{V_2} \rangle) \geq 3$. Therefore, $\dim \langle \mathbf{p}'_{V_1}, \mathbf{p}'_{V_2} \rangle = \dim \langle \mathbf{p}'_{V_1} \rangle + \dim \langle \mathbf{p}'_{V_2} \rangle - \dim(\langle \mathbf{p}'_{V_1} \rangle \cap \langle \mathbf{p}'_{V_2} \rangle) \leq 3 + 4 - 3 = 4$.

Assume now that (say) $w'_{13} = 0$, $w'_{24}, w'_{8,10} \neq 0$. Then $\dim \langle \mathbf{p}'_{V_2} \rangle \leq 3$ (using the equilibrium condition at node 1). As $w'_{24}, w'_{8,10} \neq 0$, we know that $p'_4, p'_8 \in \langle \mathbf{p}'_{V_2} \rangle$. Hence $\dim(\langle \mathbf{p}'_{V_1} \rangle \cap \langle \mathbf{p}'_{V_2} \rangle) \geq 2$ and thus $\dim \langle \mathbf{p}'_{V_1}, \mathbf{p}'_{V_2} \rangle \leq 3 + 3 - 2 = 4$.

Assume now (say) that $w'_{13} = w'_{24} = 0$, $w'_{8,10} \neq 0$. Then the equilibrium conditions at nodes 1 and 2 imply that $\dim \langle \mathbf{p}'_{V_2} \rangle \leq 2$, contradicting (5.11).

Finally assume now that $w'_{13} = w'_{24} = w'_{8,10} = 0$. Then $\dim \langle \mathbf{p}'_{V_2} \rangle = 2$, contradicting again (5.11).

6

Relation of gd(·) with other graph parameters

Our goal in this chapter is to investigate the links between the Gram dimension of a graph and two other graphs parameters that have been studied in the literature. The first one is the *Euclidean dimension* of a graph introduced by Belk and Connelly in [25, 26]. The second one is the graph parameter $\nu^{=}(\cdot)$ introduced and studied by van der Holst in [126, 129].

The content of this chapter is based on joint work with M. Laurent [83].

6.1 Relation with Euclidean graph realizations

We start the discussion with some definitions. Recall that a matrix $D = (d_{ij}) \in \mathscr{S}^n$ is a *Euclidean distance matrix* (EDM) if there exist vectors $p_1, \ldots, p_n \in \mathbb{R}^k$ (for some $k \geq 1$) such that $d_{ij} = \|p_i - p_j\|^2$ for all $i, j \in [n]$. Then EDM_n denotes the cone of all $n \times n$ Euclidean distance matrices and, for a graph $G = ([n], E)$, $\mathrm{EDM}(G) = \pi_E(\mathrm{EDM}_n)$ is the set of G-partial matrices that can be completed to a Euclidean distance matrix.

6.1.1 Definition. *Given a graph $G = ([n], E)$ and $d \in \mathbb{R}^E_+$, a Euclidean (distance) representation of d in $\mathbb{R}^k$ consists of a set of vectors $p_1, \ldots, p_n \in \mathbb{R}^k$ such that*

$$\|p_i - p_j\|^2 = d_{ij} \ \forall ij \in E.$$

Then, $\mathrm{ed}(G, d)$ *is the smallest integer k for which d has a Euclidean representation in $\mathbb{R}^k$ and the graph parameter* $\mathrm{ed}(G)$ *is defined as*

$$\mathrm{ed}(G) = \max_{d \in \mathrm{EDM}(G)} \mathrm{ed}(G, d). \tag{6.1}$$

Following [25, 26], a graph G satisfying $\mathrm{ed}(G) \leq k$ is called *k-realizable*.

It is known that the parameter ed(·) is minor monotone [26]. Hence, for any fixed integer $k \geq 1$, the graphs satisfying $\mathrm{ed}(G) \leq k$ can be characterized by a finite list of minimal forbidden minors. For $k \leq 2$ the only forbidden minor is K_{k+2}. For the case $k = 3$, the list of forbidden minors was identified by Belk and Connelly [25, 26].

6.1.2 Theorem. *[25, 26] For a graph G, $\mathrm{ed}(G) \leq 3$ if and only if G does not have K_5 and $K_{2,2,2}$ as minors.*

The crux of the proof of Theorem 6.1.2 is to prove that if a graph G has no K_5 and $K_{2,2,2}$ minors then $\mathrm{ed}(G) \leq 3$. As we will see, using the forbidden minor characterization of graphs with Gram dimension at most four (cf. Theorem 5.3.2) we can recover Theorem 6.1.2. To this end, we have to establish some connections between the graphs parameters ed(·) and gd(·).

There is a well known correspondence between psd and EDM completions (for details and references see, e.g., [44]). Namely, for a graph G, let ∇G denote its *suspension graph*, obtained by adding a new node (the *apex* node, denoted by 0), adjacent to all nodes of G. Consider the one-to-one map $\phi : \mathbb{R}^{V \cup E(G)} \mapsto \mathbb{R}^{E(\nabla G)}$, which maps $x \in \mathbb{R}^{V \cup E(G)}$ to $d = \phi(x) \in \mathbb{R}^{E(\nabla G)}$ defined by

$$d_{0i} = x_{ii} \ (i \in [n]), \quad d_{ij} = x_{ii} + x_{jj} - 2x_{ij} \ (ij \in E(G)). \tag{6.2}$$

Then, for an element $x \in \mathscr{S}_+(G)$, the vectors $u_1, \ldots, u_n \in \mathbb{R}^k$ form a Gram representation of x if and only if the vectors $u_0 = 0, u_1, \ldots, u_n$ form a Euclidean representation of $d = \phi(x)$ in $\mathbb{R}^k$. This shows:

6.1.3 Lemma. *For a graph $G = (V, E)$ and a vector $x \in \mathscr{S}_+(G)$, we have that $\mathrm{gd}(G, x) = \mathrm{ed}(\nabla G, \phi(x))$ and thus $\mathrm{gd}(G) = \mathrm{ed}(\nabla G)$.*

Recall that for any graph G we have that $\mathrm{gd}(\nabla G) = \mathrm{gd}(G) + 1$; cf. Lemma 5.2.9. We do not know whether the analogous property is true for the graph parameter ed(·). On the other hand, the following partial result holds, whose proof follows from discussions with Lex Schrijver.

6.1.4 Theorem. *For a graph G, $\mathrm{ed}(\nabla G) \geq \mathrm{ed}(G) + 1$.*

Proof. Set $\mathrm{ed}(\nabla G) = k$; we show $\mathrm{ed}(G) \leq k - 1$. We may assume that G is connected (else deal with each connected component separately). Let $d \in \mathrm{EDM}(G)$ and let $p_1 = 0, p_2, \ldots, p_n$ be a Euclidean representation of d in $\mathbb{R}^h$ ($h \geq 1$). Extend the p_i's to vectors $\widehat{p_i} = (p_i, 0) \in \mathbb{R}^{h+1}$ by appending an extra coordinate equal to zero, and set $\widehat{p}_0(t) = (0, t) \in \mathbb{R}^{h+1}$ where t is any positive real scalar. Now consider the distance $\widehat{d}(t) \in \mathrm{EDM}(\nabla G)$ with Euclidean representation $\widehat{p_0}(t), \widehat{p_1}, \ldots, \widehat{p_n}$.

As $\mathrm{ed}(\nabla G) = k$, there exists another Euclidean representation of $\widehat{d}(t)$ by vectors $q_0(t), q_1(t), \ldots, q_n(t)$ lying in $\mathbb{R}^k$. Without loss of generality, we can assume that $q_0(t) = \widehat{p_0}(t) = (0, t)$ and $q_1(t)$ is the zero vector; for $i \in [n]$, write $q_i(t) = (u_i(t), a_i(t))$, where $u_i(t) \in \mathbb{R}^{k-1}$ and $a_i(t) \in \mathbb{R}$. Then $\|q_i(t)\| = \|\widehat{p_i}\| = \|p_i\|$ whenever node i is adjacent to node 1 in G. As the graph G is connected, this implies that, for any $i \in [n]$, the scalars $\|q_i(t)\|$ ($t \in \mathbb{R}_+$) are bounded. Therefore there exists a sequence $t_m \in \mathbb{R}_+$ ($m \in \mathbb{N}$) converging to $+\infty$ and for which the sequence $(q_i(t_m))_m$ has a limit. Say $q_i(t_m) = (u_i(t_m), a_i(t_m))$ converges to $(u_i, a_i) \in$

$\mathbb{R}^k$ as $m \to +\infty$, where $u_i \in \mathbb{R}^{k-1}$ and $a_i \in \mathbb{R}$. The condition $\|q_0(t) - q_i(t)\|^2 = \widehat{d}(t)_{0i}$ implies that $\|u_i(t)\|^2 + (a_i(t) - t)^2 = \|p_i\|^2 + t^2$ and thus

$$a_i(t_m) = \frac{a_i^2(t_m) + \|u_i(t_m)\|^2 - \|p_i\|^2}{2t_m} \quad \forall m \in \mathbb{N}.$$

Taking the limit as $m \to \infty$ we obtain that $\lim_{m\to\infty} a_i(t_m) = 0$ and thus $a_i = 0$. Then, for $ij \in E$, $d_{ij} = \widehat{d}(t_m)_{ij} = \|(u_i(t_m), (a_i(t_m)) - (u_j(t_m), a_j(t_m))\|^2$ and taking the limit as $m \to +\infty$ we obtain that $d_{ij} = \|u_i - u_j\|^2$. This shows that the vectors $u_1, \ldots, u_n$ form a Euclidean representation of d in $\mathbb{R}^{k-1}$. $\square$

Combining Lemma 6.1.3 with Theorem 6.1.4 we obtain an inequality relating the parameters ed$(\cdot)$ and gd$(\cdot)$.

6.1.5 Theorem. *For any graph G we have that* $\mathrm{ed}(G) \leq \mathrm{gd}(G) - 1$.

Combining Theorem 6.1.5 with Theorem 5.3.2 we can recover sufficiency in Theorem 6.1.2.

6.1.6 Corollary. *For a graph G, if G has no K_5 and $K_{2,2,2}$ minors then* $\mathrm{ed}(G) \leq 3$.

We conclude this section with some well-known facts concerning the complexity of deciding whether a given vector $d \in \mathbb{Q}_+^E$ admits a Euclidean representation in $\mathbb{R}^k$. These results will be useful in Chapter 7. Formally, for fixed $k \geq 1$, we consider the following problem:

Given a graph $G = (V, E)$ and $d \in \mathbb{Q}_+^E$, decide whether $\mathrm{ed}(G, d) \leq k$.

Using a reduction from the 3SAT problem, Saxe obtained the following:

6.1.7 Theorem. *[118] For any fixed $k \geq 1$, deciding whether* $\mathrm{ed}(G, d) \leq k$ *is* NP-*hard, already when restricted to weights $d \in \{1, 2\}^E$.*

6.2 Relation with the graph parameter $\nu^=(\cdot)$

In this section we investigate the relation between gd$(\cdot)$ and the graph parameter $\nu^=(\cdot)$ introduced in [126, 129]. Recall that the corank of a matrix $M \in \mathbb{R}^{n\times n}$ is the dimension of its kernel. For a graph $G = (V = [n], E)$ consider the cone

$$\mathscr{C}(G) = \{M \in \mathscr{S}_+^n : M_{ij} = 0 \text{ for all distinct } i, j \in V \text{ with } ij \notin E\},$$

which, as is well-known, can be seen as the dual cone of the cone $\mathscr{S}_+(G)$.

We now introduce the graph parameter $\nu^=(\cdot)$.

6.2.1 Definition. *For a graph $G = (V = [n], E)$, the parameter $\nu^=(G)$ is defined as the maximum corank of a matrix $M \in \mathscr{C}(G)$ satisfying the SAP, i.e.,*

$$X \in \mathscr{S}^n, \quad MX = 0,\ X_{ii} = 0\ \forall i \in V,\ X_{ij} = 0\ \forall ij \in E \implies X = 0.$$

The study of the graph parameter $\nu^=(\cdot)$ is motivated by its relevance to the celebrated graph parameter $\mu(\cdot)$, introduced by Colin de Verdière [42]; cf. Section 3.4.

It was shown in [126, 129] that $\nu^=(\cdot)$ is a minor monotone graph parameter. Hence, for any fixed integer $k \geq 1$, the graphs with $\nu^=(G) \leq k$ can be characterized by a finite family of minimal forbidden minors. For $k \leq 3$ the only forbidden minor is K_{k+1}. The list of forbidden minors for $k = 4$ was determined in [126, 129].

6.2.2 Theorem. *[126, 129] For a graph G, $\nu^=(G) \leq 4$ if and only if G does not have K_5 and $K_{2,2,2}$ as minors.*

The heart of the proof of Theorem 6.2.2 is to show that any graph which is K_5 and $K_{2,2,2}$-minor free satisfies $\nu^=(G) \leq 4$. In the next theorem we establish a relation between the two parameters $\mathrm{gd}(\cdot)$ and $\nu^=(\cdot)$ which allows us to derive sufficiency in Theorem 6.2.2 from the characterization of graphs with Gram dimension at most 4; cf. Theorem 5.3.2.

6.2.3 Theorem. *For any graph G, $\mathrm{gd}(G) \geq \nu^=(G)$.*

Proof. Let $k = \nu^=(G)$ be attained by some matrix $M \in \mathcal{S}^n_+$. By the spectral theorem we can write $M = \sum_{i=1}^n \lambda_i v_i v_i^\mathsf{T}$, where $\lambda_i \geq 0$, $\{v_1, \ldots, v_n\}$ is an orthonormal base of eigenvectors of M, and $\{v_1, \ldots, v_k\}$ spans the kernel of M. Consider the matrix $X = \sum_{i=1}^k v_i v_i^\mathsf{T}$ and its projection $a = \pi_{VE}(X) \in \mathcal{S}_+(G)$. By construction, $\mathrm{rank} X = k$. Hence it is enough to show that a has a unique psd completion, which will imply $\mathrm{gd}(G) \geq \mathrm{gd}(G, a) = k$.

For this let $Y \in \mathcal{S}^n_+$ be another psd completion of a. Hence the matrix $X - Y$ has zero entries at all positions $ij \in V \cup E$. Since the matrix M has zero entries at all off-diagonal positions corresponding to non-edges of G, we deduce that $\langle M, X - Y\rangle = 0$. On the other hand, $\langle M, X\rangle = \sum_{i=1}^k \lambda_i v_i^\mathsf{T} M v_i = 0$. Therefore, $\langle M, Y\rangle = 0$. As M, X, Y are psd, the conditions $\langle M, X\rangle = \langle M, Y\rangle = 0$ imply that $MX = MY = 0$ and thus $M(X - Y) = 0$. Now we can apply the assumption that the matrix M satisfies the Strong Arnold Property and deduce that $X = Y$. □

Combining Theorem 6.2.3 with Theorem 5.3.2 we can recover sufficiency (which is the difficult direction) in Theorem 6.2.2.

6.2.4 Corollary. *If G does not have K_5 and $K_{2,2,2}$ as minors then $\nu^=(G) \leq 4$.*

In view of Theorem 6.2.3 one can ask whether the parameters $\mathrm{gd}(\cdot)$ and $\nu^=(\cdot)$ coincide or not. This question remains an open problem. As a first step in understanding the exact relation between these two parameters, we now derive a characterization of the parameter $\nu^=(\cdot)$ in terms of the maximum Gram dimension of some G-partial psd matrices satisfying a certain nondegeneracy property. For the remainder of this section, with a vector $a \in \mathcal{S}_+(G)$ we associate the following pair of primal and dual semidefinite programs:

$$\sup_X \left\{0 : \langle E_{ij}, X\rangle = a_{ij} \text{ for } \{i,j\} \in V \cup E, \text{ and } X \succeq 0\right\}, \tag{P_a}$$

$$\inf_{y,Z} \{ \sum_{\{i,j\} \in V \cup E} y_{ij} a_{ij} : \sum_{\{i,j\} \in V \cup E} y_{ij} E_{ij} = Z \succeq 0\}. \tag{D_a}$$

Notice that, for any $a \in \mathcal{S}_+(G)$, the primal program (P_a) is feasible and the dual program (D_a) is strictly feasible. Thus there is no duality gap.

6.2.5 Definition. *For a graph G, let $\mathcal{D}(G)$ denote the set of partial matrices $a \in \mathcal{S}_+(G)$ for which the semidefinite program (D_a) has a nondegenerate optimal solution.*

We can now reformulate the parameter $\nu^=(\cdot)$ as the maximum Gram dimension of a partial matrix in $\mathcal{D}(G)$.

6.2.6 Theorem. *For any graph G we have that*

$$\nu^{=}(G) = \max_{a \in \mathscr{D}(G)} \mathrm{gd}(G,a).$$

Proof. Suppose that $\max_{a \in \mathscr{D}(G)} \mathrm{gd}(G,a) = \mathrm{gd}(G,a^*)$. As $a^* \in \mathscr{D}(G)$ it follows that (D_{a^*}) has a nondegenerate optimal solution which we denote by M. Then, Theorem 3.3.4 implies that (P_{a^*}) has a unique solution which we denote by A. Notice that the matrix A is the unique psd completion of the partial matrix $a^* \in \mathscr{S}_+(G)$ which implies that $\mathrm{gd}(G,a^*) = \mathrm{rank} A$. Moreover, as A and M are a pair of primal dual optimal solutions we have that $AM = 0$ which implies that $\mathrm{corank}\, M \geq \mathrm{rank} A$. By Theorem 3.4.3, the fact that M is a nondegenerate solution of (D_{a^*}) implies that M satisfies the SAP. Consequently, the matrix M is feasible for $\nu^{=}(G)$ and it follows that $\nu^{=}(G) \geq \max_{a \in \mathscr{D}(G)} \mathrm{gd}(G,a)$.

For the other direction, assume $\nu^{=}(G) = \mathrm{corank}\, M = d$ where $M \in \mathscr{C}(G)$ and M satisfies the SAP. Let $P \in \mathbb{R}^{n \times d}$ be a matrix whose columns form a basis for $\mathrm{Ker}\, M$ and consider the partial matrix $a \in \mathscr{S}_+(G)$ defined as $a_{ij} = (PP^\mathsf{T})_{ij}$ for every $\{i,j\} \in V \cup E$. As $\langle M, PP^\mathsf{T} \rangle = 0$ it follows that M is a dual nondegenerate optimal solution for (D_a) and thus $a \in \mathscr{D}(G)$. Additionally, as $\mathrm{corank}\, M = \mathrm{rank}\, PP^\mathsf{T}$, M and PP^T are a pair of strict complementary optimal solutions for (P_a) and (D_a), respectively. Then, Theorem 3.3.6 implies that the matrix PP^T is the unique optimal solution of (P_a) and thus $\mathrm{gd}(G,a) = \mathrm{rank}\, PP^\mathsf{T} = \mathrm{corank}\, M = \nu^{=}(G)$. $\square$

As a direct corollary we can reformulate the problem of deciding whether the parameters $\mathrm{gd}(\cdot)$ and $\nu^{=}(\cdot)$ coincide as follows.

6.2.7 Corollary. *For any graph G, we have that* $\mathrm{gd}(G) \geq \nu^{=}(G)$. *Moreover, equality holds if and only if there exists some $a \in \mathscr{D}(G)$ for which* $\mathrm{gd}(G) = \mathrm{gd}(G,a)$.

Closing this section we show that Theorem 6.2.3 implies that the Gram dimension is unbounded for the class of planar graphs. Colin de Verdière [43] studies the graph parameter $\nu(G)$, defined as the maximum corank of a matrix M satisfying the strong Arnold property and such that, for any $i,j \in V$, $M_{ij} = 0 \iff ij \notin E$. In particular it is shown in [43, Theorem 6] shows that $\nu(G)$ is unbounded for the class of planar graphs. As $\nu(G) \leq \nu^{=}(G) \leq \mathrm{gd}(G)$, we obtain as a direct application:

6.2.8 Corollary. *The parameter* $\mathrm{gd}(\cdot)$ *is unbounded for the class of planar graphs.*

An explicit family of planar graphs for which the Gram dimension is unbounded is described and studied in Section 10.1.4.

7 The complexity of the low rank psd matrix completion problem

In this chapter we address various complexity aspects of the positive semidefinite matrix completion problem. In particular, we consider the decision problem where we are given as input a rational G-partial matrix and the goal is to verify whether it has a positive semidefinite completion of rank at most k, for some fixed integer $k \geq 1$. Our main result is that this problem is NP-hard for every fixed $k \geq 2$. Additionally we consider the membership problem in the convex hull of $\mathscr{E}_k(G)$ and show it is NP-hard for any fixed $k \geq 2$.

The content of this chapter are based on joint work with M. E.-Nagy and M. Laurent [47].

7.1 Membership in the rank constrained elliptope $\mathscr{E}_k(G)$

Given an integer $k \geq 1$, consider the set

$$\mathscr{E}_{n,k} = \{X \in \mathscr{S}^n_+ : \text{rank} X \leq k,\ X_{ii} = 1\ (i \in [n])\},$$

known as the *rank constrained elliptope*. Moreover, for a graph $G = ([n], E)$, let $\mathscr{E}_k(G)$ denote the projection of $\mathscr{E}_{n,k}$ onto the coordinates indexed by the edges of the graph G, i.e.,

$$\mathscr{E}_k(G) = \pi_E(\mathscr{E}_{n,k}),$$

where $\pi_E : \mathscr{S}^n \mapsto \mathbb{R}^E$ with $X \mapsto (X_{ij})_{ij \in E}$. Our main goal in this section is to understand the complexity of testing membership in $\mathscr{E}_k(G)$.

Throughout this section, for a vector $x \in \mathbb{R}^E$, we use the shorthand notation $\text{gd}(G, x)$ in place of $\text{gd}(G, (e, x))$. Then $\text{gd}(G, x)$ is equal to the smallest integer $k \geq 1$ for which x admits a Gram representation by unit vectors in $\mathbb{R}^k$ (recall Remark 5.2.3). Furthermore, the points x in $\mathscr{E}_k(G)$ correspond precisely to those vectors $x \in \mathbb{R}^E$ that admit a Gram representation by unit vectors in $\mathbb{R}^k$; that is:

$$x \in \mathscr{E}_k(G) \Longleftrightarrow \mathrm{gd}(G,x) \le k.$$

Recall that the elements of $\mathscr{E}(G)$ can be seen as the G-partial positive semidefinite matrices, that can be completed to full correlation matrices. Hence, for fixed $k \ge 1$, the membership problem in $\mathscr{E}_k(G)$ is the problem of deciding whether a given G-partial matrix has a psd completion of rank at most k. Using the notion of Gram dimension this can be equivalently formalized as:

Given a graph $G = (V,E)$ and $x \in \mathbb{Q}^E$, decide whether $\mathrm{gd}(G,x) \le k$.

For $k = 1$, $x \in \mathscr{E}_1(G)$ if and only if $x \in \{\pm 1\}^E$ corresponds to a cut of G, and this can be decided in polynomial time. In the following sections we show that the membership problem in $\mathscr{E}_k(G)$ is NP-hard for any fixed $k \ge 2$. It turns out that we have to use different reductions for the cases $k \ge 3$ and $k = 2$.

7.1.1 The case $k \ge 3$

First we consider the problem of testing membership in $\mathscr{E}_k(G)$ when $k \ge 3$. We show that this is an NP-hard problem, already when $G = \nabla^{k-3}H$ is the suspension of a planar graph H and $x = \mathbf{0}$, the vector with zero entries at all edges (and ones at entries corresponding to nodes). Recall that $\nabla^p G$ denotes the graph obtained from G, by iteratively applying the suspension operation p times.

For the hardness reduction, the key idea is to relate the parameter $\mathrm{gd}(G,\mathbf{0})$ to the chromatic number of the graph. Notice that $\mathrm{gd}(G,\mathbf{0})$ is equal to the smallest dimension where the graph G admits an orthonormal representation. This quantity was introduced and studied by Lovász as an upper bound to the theta number.

7.1.1 Theorem. *[87] For any graph G we have that*

$$\vartheta(G) \le \mathrm{gd}(\overline{G},\mathbf{0}).$$

Proof. We use the following formulation for the ϑ number:

$$\begin{aligned} \vartheta(G) = \min\ & t \\ \text{s.t. } & X_{00} = t \\ & X_{0i} = X_{ii} = 1 \ (0 \le i \le n) \\ & X_{ij} = 0 \ (1 \le i \le n) \text{ and } ij \ne E. \end{aligned} \tag{7.1}$$

Assuming $\mathrm{gd}(G,\mathbf{0}) = d$, there exist vectors $p_1,\dots,p_n \in \mathbb{R}^d$ satisfying $p_i^{\mathsf{T}} p_j = 0$ for all $ij \notin E$. Consider the matrix $X^* = \mathrm{Gram}(I_d, p_1p_1^{\mathsf{T}},\dots,p_np_n^{\mathsf{T}})$. Then, X^* is feasible for (7.1) and its objective value is equal to d. □

It is easy to verify that

$$\mathrm{gd}(G,\mathbf{0}) \le \chi(G), \tag{7.2}$$

with equality if $\chi(G) \le 2$ (i.e., if G is a bipartite graph). For $k \ge 3$ the inequality (7.2) can be strict. This is the case, e.g., for orthogonality graphs of Kochen-Specker sets (see [60]).

Given a set of vectors $S = \{s_1,\dots,s_k\} \subseteq \mathbb{C}^n$, its *orthogonality graph* is the graph with vertex set $[k]$, where two nodes $i,j \in [k]$ are adjacent if and only if $\langle s_i, s_j\rangle = 0$.

7.1.2 Definition. *A set $S \subseteq \mathbb{C}^n$ is called a* Kochen-Specker (KS) set *if there does not exist a function $f : \mathbb{C}^n \mapsto \{0,1\}$ satisfying*

$$\sum_{i \in B} f(i) = 1, \textit{ for every orthonormal base } B \textit{ of } \mathbb{C}^n \textit{ with } B \subseteq S.$$

Proving the existence of KS sets is a challenging task. We are interested in the existence of KS sets in $\mathbb{R}^3$. The first such example is a family of 117 vectors given in [69] and the smallest known KS set in $\mathbb{R}^3$ contains 31 vectors [102].

7.1.3 Theorem. *[60] Let $S \subseteq \mathbb{R}^3$ be a Kochen-Specker set and let G_S be its orthogonality graph. Then* $\mathrm{gd}(G_S, \mathbf{0}) < \chi(G_S)$.

Proof. By definition we have that $V(G_S) = \{i : i \in S\}$ and $ij \in E(G_S)$ if and only if $\langle i, j\rangle = 0$. Clearly, $\mathrm{gd}(G_s, \mathbf{0}) \leq 3$. Assume for contradiction that $\chi(G_S) \leq 3$ and consider any function $f : \mathbb{C}^3 \mapsto \{0,1\}$ that assigns the value 1 to the first color class and the value 0 to the other color classes. Then if B is an orthonormal basis of $\mathbb{C}^3$ contained in S, by definition of G_S, the set $\{i : i \in B\}$ forms a 3-clique in G_S and thus exactly one of these three nodes corresponds to the first class. This shows that $\sum_{i\in B} f(i) = 1$ a contradiction. $\square$

However, Peeters [101, Theorem 3.1] gives a polynomial time reduction of the problem of deciding 3-colorability of a graph to that of deciding $\mathrm{gd}(G, \mathbf{0}) \leq 3$. Namely, given a graph G, he constructs (in polynomial time) a new graph G' having the property that

$$\chi(G) \leq 3 \iff \chi(G') \leq 3 \iff \mathrm{gd}(G', \mathbf{0}) \leq 3. \tag{7.3}$$

The graph G' is obtained from G by adding for each pair of distinct nodes $i, j \in V$ the gadget graph H_{ij} shown in Figure 7.1. Moreover, using a more involved con-

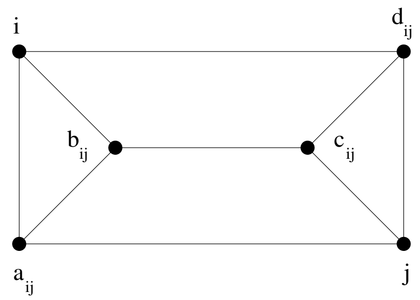

Figure 7.1: The gadget graph H_{ij}.

struction, Peeters [100] constructs (in polynomial time) from any planar graph G a new *planar* graph G' satisfying (7.3). As the problem of deciding 3-colorability is NP-complete already for planar graphs (see [125]), we have the following result.

7.1.4 Theorem. *[100] It is* NP*-hard to decide whether* $\mathrm{gd}(G, \mathbf{0}) \leq 3$*, already for the class of planar graphs.*

This hardness result can be extended to any fixed $k \geq 3$ using the suspension operation on graphs. Indeed, it is an easy observation that

$$\mathrm{gd}(\nabla^p G, \mathbf{0}) = \mathrm{gd}(G, \mathbf{0}) + p. \tag{7.4}$$

Theorem 7.1.4 combined with equation (7.4) implies:

7.1.5 Theorem. *Fix $k \geq 3$. It is* NP*-hard to decide whether* $\mathrm{gd}(G, \mathbf{0}) \leq k$*, already for graphs of the form $G = \nabla^{k-3}H$ where H is a planar graph.*

As an application we can recover the complexity result of Saxe from Theorem 6.1.7 for the case $k \geq 3$.

7.1.6 Corollary. *For fixed $k \geq 3$, it is an* NP*-hard to decide whether* $\mathrm{ed}(G, d) \leq k$*, already when $G = \nabla^{k-2}H$ with H planar and d is $\{1,2\}$-valued (more precisely, all edges adjacent to a given apex node have weight 1 and all other edges have weight 2).*

Proof. This follows directly from Lemma 6.1.3 combined with Theorem 7.1.5: By Lemma 6.1.3, $\mathrm{gd}(\nabla^{k-3}H, \mathbf{0}) = \mathrm{ed}(\nabla^{k-2}H, \phi(\mathbf{0}))$ and observe that the image $d = \phi(\mathbf{0})$ of the zero vector under the covariance map ϕ (recall (6.2)) satisfies: $d_{0i} = 1$ and $d_{ij} = 2$ for all nodes i, j of $\nabla^{k-3}H$. □

7.1.2 The case $k = 2$

In this section we show NP-hardness of testing membership in $\mathcal{E}_2(G)$. Our strategy to show this result is as follows: Given a graph $G = (V, E)$ with edge weights $d \in \mathbb{R}^E_+$, define the new edge weights $x = \cos d \in \mathbb{R}^E$. We show a close relationship between the two problems of testing whether $\mathrm{ed}(G, d) \leq 1$, and whether $\mathrm{gd}(G, x) \leq 2$ (or, equivalently, $x \in \mathcal{E}_2(G)$). More precisely, we show that each of these two properties can be characterized in terms of the existence of a ± 1-signing of the edges of G satisfying a suitable "flow conservation" type property; moreover, both are equivalent when the edge weights d are small enough.

As a motivation, let us consider first the case when $G = C_n$ is a circuit of length n. Say, weight d_i (resp., $x_i = \cos d_i$) is assigned to the edge $(i, i+1)$ for $i \in [n]$ (where indices are taken modulo n). Then the following property holds:

$$\mathrm{ed}(C_n, d) \leq 1 \iff \exists \epsilon \in \{\pm 1\}^n \text{ such that } \epsilon^{\mathsf{T}} d = 0. \tag{7.5}$$

This is the key fact used by Saxe [118] for showing NP-hardness of the problem of testing $\mathrm{ed}(C_n, d) \leq 1$ by reducing the Partition problem for $d = (d_1, \cdots, d_n) \in \mathbb{Z}^n_+$ to it. Recall that in Lemma 5.2.11 we showed the analogous property for the Gram dimension:

$$\mathrm{gd}(C_n, \cos d) \leq 2 \iff \exists \epsilon \in \{\pm 1\}^n \text{ such that } \epsilon^{\mathsf{T}} d \in 2\pi\mathbb{Z}. \tag{7.6}$$

We now observe that these two characterizations extend to an arbitrary graph G. To formulate the result we need to fix an (arbitrary) orientation $\widetilde{G}$ of G. Let $P = (u_0, u_1, \cdots, u_{k-1}, u_k)$ be a walk in G, i.e., $\{u_i, u_{i+1}\} \in E$ for all $0 \leq i \leq k-1$. Recall that in a walk repetition of vertices is allowed; the walk P is said to be *closed* when $u_0 = u_k$. For $\epsilon \in \{\pm 1\}^E$, we define the following weighted sum along the edges of P:

$$\phi_{d,\epsilon}(P) = \sum_{i=0}^{k-1} d_{u_i, u_{i+1}} \epsilon_{u_i u_{i+1}} \eta_i, \tag{7.7}$$

setting $\eta_i = 1$ if the edge $\{u_i, u_{i+1}\}$ is oriented in $\widetilde{G}$ from u_i to u_{i+1} and $\eta_i = -1$ otherwise.

7.1.7 Lemma. *Consider a graph $G = (V, E)$ with edge weights $d \in \mathbb{R}_+^E$ and fix an orientation $\widetilde{G}$ of G. The following assertions are equivalent.*

(i) $\mathrm{ed}(G, d) \leq 1$.

(ii) *There exists an edge-signing $\epsilon \in \{\pm 1\}^E$ for which the function $\phi_{d,\epsilon}$ from (7.7) satisfies: $\phi_{d,\epsilon}(C) = 0$ for all closed walks C of G (equivalently, for all circuits of G).*

Proof. Assume that (i) holds. Let $f : V \to \mathbb{R}$ satisfying $|f(u) - f(v)| = d_{uv}$ for all $\{u, v\} \in E$. If the edge $\{u, v\}$ is oriented from u to v in $\widetilde{G}$, let $\epsilon_{uv} \in \{\pm 1\}$ such that $f(v) - f(u) = d_{uv}\epsilon_{uv}$. This defines an edge-signing $\epsilon \in \{\pm 1\}^E$; we claim that (ii) holds for this edge-signing. For this, pick a circuit $C = (u_0, u_1, \cdots, u_k = u_0)$ in G. By construction of the edge-signing, the term $\epsilon_{u_i u_{i+1}} d_{u_i u_{i+1}} \eta_i$ is equal to $f(u_{i+1}) - f(u_i)$ for all $0 \leq i \leq k - 1$, where indices are taken modulo k. This implies that $\phi_{d,\epsilon}(C) = \sum_{i=0}^{k-1}(f(u_{i+1}) - f(u_i)) = 0$ and thus (ii) holds. Conversely, assume (ii) holds. We may assume that G is connected (else apply the following to each connected component). Fix an arbitrary node $u_0 \in V$. We define the function $f : V \to \mathbb{R}$ by setting $f(u_0) = 0$ and, for $u \in V \setminus \{u_0\}$, $f(u) = \phi_{d,\epsilon}(P)$ where P is any walk from u_0 to u. It is easy to verify that since (ii) holds this definition does not depend on the choice of P. We claim that f is a Euclidean embedding of (G, d) into $\mathbb{R}$. For this, pick an edge $\{u, v\} \in E$; say, it is oriented from u to v in $\widetilde{G}$. Pick a walk P from u_0 to u, so that $Q = (P, v)$ is a walk from u_0 to v. Then, $f(u) = \phi_{d,\epsilon}(P)$, $f(v) = \phi_{d,\epsilon}(Q) = \phi_{d,\epsilon}(P) + d_{uv}\epsilon_{uv} = f(u) + d_{uv}\epsilon_{uv}$, which implies that $|f(v) - f(u)| = d_{uv}$. □

Next we prove the analogous result for the Gram setting.

7.1.8 Lemma. *Consider a graph $G = (V, E)$ with edge weights $d \in \mathbb{R}_+^E$ and fix an orientation $\widetilde{G}$ of G. The following assertions are equivalent.*

(i) $\mathrm{gd}(G, \cos d) \leq 2$.

(ii) *There exists an edge-signing $\epsilon \in \{\pm 1\}^E$ for which the function $\phi_{d,\epsilon}$ from (7.7) satisfies: $\phi_{d,\epsilon}(C) \in 2\pi\mathbb{Z}$ for all closed walks C of G (equivalently, for all circuits of G).*

Proof. Assume (i) holds. Then, there exists a labeling of the nodes $u \in V$ by unit vectors $g(u) = (\cos f(u), \sin f(u))$ where $f(u) \in [0, 2\pi]$ such that for any edge $\{u, v\} \in E$, we have $\cos d_{uv} = g(u)^{\mathsf{T}} g(v) = \cos(f(u) - f(v))$. If $\{u, v\}$ is oriented from u to v, define $\epsilon_{uv} \in \{\pm 1\}$ such that $\epsilon_{uv} d_{uv} = f(v) - f(u) + 2\pi\mathbb{Z}$. This defines an edge-signing $\epsilon \in \{\pm 1\}^E$ which satisfies (ii) (same argument as in the proof of Lemma 7.1.7).

Conversely, assume (ii) holds. Analogously to the proof of Lemma 7.1.7, fix a node $u_0 \in V$ and consider the unit vectors $g(u_0) = (1, 0)$ and

$$g(u) = (\cos(\phi_{d,\epsilon}(P_u)), \sin(\phi_{d,\epsilon}(P_u))),$$

where P_u is a walk from $u_0 \in V$ to $u \in V \setminus \{u_0\}$. Pick an edge $\{u, v\} \in E$ and say it is oriented from u to v in $\widetilde{G}$. Pick a walk P from u_0 to u, so that $Q = (P, v)$ is a walk from u_0 to v. Then $g(u)^{\mathsf{T}} g(v) = \cos(\phi_{d,\epsilon}(P) - \phi_{d,\epsilon}(Q)) = \cos(\epsilon_{uv} d_{uv} + 2\pi\mathbb{Z}) = \cos d_{uv}$. □

7.1.9 Corollary. *Consider a graph $G = (V, E)$ with edge weights $d \in \mathbb{R}^E_+$ satisfying $d(E) < 2\pi$. Then,* $\mathrm{ed}(G, d) \leq 1$ *if and only if* $\mathrm{gd}(G, \cos d) \leq 2$.

Proof. Note that if C is a circuit of G, then $\phi_{d,\epsilon}(C) \in 2\pi\mathbb{Z}$ implies $\phi_{d,\epsilon}(C) = 0$, since $|\phi_{d,\epsilon}(C)| \leq d(E) < 2\pi$. The result now follows directly by applying Lemmas 7.1.7 and 7.1.8. □

We can now show NP-hardness of testing membership in the rank constrained elliptope $\mathscr{E}_2(G)$. For this we use the result of Theorem 6.1.7 for the case $k = 1$: Given edge weights $d \in \{1, 2\}^E$, it is NP-hard to decide whether $\mathrm{ed}(G, d) \leq 1$. Notice that Corollary 7.1.9 does not imply that testing membership in $\mathscr{E}_2(G)$ is NP-hard since the reduction it describes involves irrational numbers; For $d \in \{1, 2\}^E$ we have that $\cos d$ is irrational.

7.1.10 Theorem. *Given a graph $G = (V, E)$ and rational edge weights $x \in \mathbb{Q}^E$, it is* NP*-hard to decide whether $x \in \mathscr{E}_2(G)$ or, equivalently,* $\mathrm{gd}(G, x) \leq 2$.

Proof. Fix edge weights $d \in \{1, 2\}^E$. We will use the fact that $\mathrm{ed}(G, d) = \mathrm{ed}(G, ad)$ for any scalar $a > 0$. Using Corollary 7.1.9 we reduce the problem of testing whether $\mathrm{ed}(G, d) \leq 1$ to the problem of testing whether $\mathrm{gd}(G, \cos(\alpha d)) \leq 2$, for some appropriately chosen parameter $\alpha > 0$.

In order to use Corollary 7.1.9 it suffices to identify a scalar $a > 0$ such that $a < 2\pi/d(E)$, $\cos a$ is rational and its bit size is polynomially bounded by the bit size of the $d's$. If such a scalar $a > 0$ exists then for $d \in \{1, 2\}^E$ we have that

$$\mathrm{ed}(G, d) \leq 1 \Longleftrightarrow \mathrm{ed}(G, ad) \leq 1 \Longleftrightarrow \mathrm{gd}(G, \cos(ad)) \leq 2,$$

where the last equivalence follows from Corollary 7.1.9. Moreover, notice that (under the aforementioned conditions on a) this reduction can be carried out in the bit model of computation in polynomial time. Indeed, as $d_e \in \{1, 2\}$ it follows that $\cos(\alpha d_e) \in \{\cos\alpha, \cos(2\alpha) = 2\cos^2\alpha - 1\} \in \mathbb{Q}$ for every $e \in E$ since by assumption we have that $\cos a \in \mathbb{Q}$. Moreover, the size of $\cos(ad_e)$ is polynomially bounded by the size of $\cos a$ and thus by the size of the input (since this is the case for $\cos a$). Lastly, we stress the fact that for the reduction we do not need the value of a itself (which could be irrational), but only the value $\cos a$.

We now identify a scalar $a > 0$ with $a \leq 1/d(E)$, $\cos a \in \mathbb{Q}$ and $|\cos a|$ is a polynomial in the size of the input. To achieve this it is enough to identify a rational function f such that

$$\cos(1/d(E)) \leq f(d) \text{ and } f(d) \in (-1, 1) \text{ for all } d = (d_e) \in \{1, 2\}^E. \tag{7.8}$$

Assuming this is possible there exists a scalar $a \in (0, \pi)$ such that $f(d) = \cos a$. Then $a \leq 1/d(E) < 2\pi/d(E)$, $\cos a \in \mathbb{Q}$ and the size of $\cos a$ is polynomial in the size of the input.

In order to find a rational function satisfying (7.8) we use the Maclaurin expansion for the cosine function. Recall that using the Maclaurin polynomial of degree 4 we have that $\cos x = T_4(x) + R_4(x)$ for every $x \in \mathbb{R}$, where $T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$ and $R_4(x) = \frac{-sin\xi}{5!}x^5$ for some ξ between x and 0. Specializing this for $x = 1/d(E)$ we have $\cos(1/d(E)) = T_4(1/d(E)) + R_4(1/d(E))$ and since $1/d(E) \in (0, 1)$ it follows that

$$\cos(1/d(E)) \leq T_4(1/d(E)).$$

Lastly, an easy calculation shows that $T_4(1/d(E)) \in (-1,1)$ and the proof is concluded. $\square$

We conclude with a remark about the complexity of the Gram dimension of weighted circuits. Consider the case when $G = C_n$ is a circuit and the edge weights $d \in \mathbb{Z}_+^{C_n}$ are integer valued. Relation (7.5) shows that $\mathrm{ed}(C_n, d) \leq 1$ if and only if the sequence $d = (d_1, \cdots, d_n)$ can be partitioned, thus showing NP-hardness of the problem of testing $\mathrm{ed}(C_n, d) \leq 1$.

As in the proof of Theorem 7.1.10 let us choose α such that $\cos\alpha, \sin\alpha \in \mathbb{Q}$ and $\alpha < 1/(\sum_{i=1}^n d_i)$; then $\cos(t\alpha) \in \mathbb{Q}$ for all $t \in \mathbb{Z}$. The analogous relation (7.6) holds, which shows that $\mathrm{gd}(C_n, \cos(\alpha d)) \leq 2$ if and only if the sequence $d = (d_1, \cdots, d_n)$ can be partitioned. However, it is not clear how to use this fact in order to show NP-hardness of the problem of testing $\mathrm{gd}(C_n, x) \leq 2$. Indeed, although all $\cos(\alpha d_i)$ are rational valued, the difficulty is that it is not clear how to compute $\cos(\alpha d_i)$ in time polynomial in the bit size of d_i (while it can be shown to be polynomial in d_i).

Finally we point out the following link to the construction of Aspnes et al. [62, §IV]. Consider the edge weights $x = \cos(\alpha d) \in \mathbb{R}^{C_n}$ for the circuit C_n and $y = \varphi(x)$ for its suspension ∇C_n, which is the wheel graph W_{n+1}. Thus $y_{0i} = 1$ and $y_{i,i+1} = 2 - 2\cos(\alpha d_i) = 4\sin^2(\alpha d_i/2)$ for all $i \in [n]$. Taking square roots we find the edge weights used in [62] to claim NP-hardness of realizing weighted wheels (that have the property of admitting unique (up to congruence) realizations in the plane). As explained in the proof of Theorem 7.1.10, if we suitably choose α we can make sure that all $\sin(\alpha d_i/2)$ be rational valued, while [62] uses real numbers. However, it is not clear how to control their bit sizes, and thus how to deduce NP-hardness.

7.2 Membership in conv $\mathscr{E}_k(G)$

In this section we investigate the complexity of optimizing a linear objective function over $\mathscr{E}_k(G)$ or, equivalently, over its convex hull conv $\mathscr{E}_k(G)$. More precisely, for any fixed $k \geq 1$ we consider the following problem:

Given a graph $G = (V, E)$ and $x \in \mathbb{Q}^E$, decide whether $x \in \mathrm{conv}\, \mathscr{E}_k(G)$.

The study of this problem is motivated by the relevance of the set conv $\mathscr{E}_k(G)$ to the MAX CUT problem and to the rank constrained Grothendieck problem (cf. Chapter 8). Indeed, for $k = 1$, conv $\mathscr{E}_1(G)$ coincides with the cut polytope of G and it is well known that linear optimization over the cut polytope is NP-hard [93]. For any $k \geq 2$, the worst case ratio of optimizing a linear function over the elliptope $\mathscr{E}(G)$ versus the rank constrained elliptope $\mathscr{E}_k(G)$ (equivalently, versus the convex hull conv $\mathscr{E}_k(G)$) is known as the *rank-k Grothendieck constant* of the graph G (cf. Section 8). It is believed that linear optimization over conv $\mathscr{E}_k(G)$ is also hard for any fixed k (cf., e.g., the quote of Lovász [90, p. 61]). We show in this section that the membership problem in conv $\mathscr{E}_k(G)$ for rational vectors is NP-hard, thus providing some evidence of hardness of optimization (cf. Theorem 7.2.2).

To prove the hardness result, the key fact is to consider the membership problem in conv $\mathscr{E}_k(G)$ for extreme points of the elliptope $\mathscr{E}(G)$. For a point $x \in \mathrm{ext}\, \mathscr{E}(G)$ we have that,

$$x \in \mathrm{conv}\, \mathscr{E}_k(G) \Longleftrightarrow x \in \mathscr{E}_k(G). \tag{7.9}$$

Our strategy for showing hardness of membership in conv$\mathscr{E}_k(G)$ is as follows: Given a graph $G = (V, E)$ and a rational vector $x \in \mathscr{E}(G)$, we construct (in polynomial time) a new graph $\widehat{G} = (\widehat{V}, \widehat{E})$ (containing G as a subgraph) and a new rational vector $\widehat{x} \in \mathbb{Q}^{\widehat{E}}$ (extending x) satisfying the following properties:

$$\widehat{x} \in \text{ext } \mathscr{E}(\widehat{G}), \tag{7.10}$$

$$x \in \mathscr{E}_k(G) \Longleftrightarrow \widehat{x} \in \mathscr{E}_k(\widehat{G}). \tag{7.11}$$

Combining these two conditions with (7.9), we deduce:

$$x \in \mathscr{E}_k(G) \Longleftrightarrow \widehat{x} \in \mathscr{E}_k(\widehat{G}) \Longleftrightarrow \widehat{x} \in \text{conv}\,\mathscr{E}_k(\widehat{G}). \tag{7.12}$$

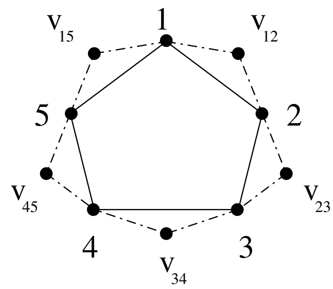

Figure 7.2: The graph $\widehat{C}_5$.

Given $G = (V, E)$, the construction of the new graph $\widehat{G} = (\widehat{V}, \widehat{E})$ is as follows: For each edge $\{i, j\}$ of G, we add a new node v_{ij}, adjacent to the two nodes i and j. Let C_{ij} denote the clique on $\{i, j, v_{ij}\}$ and set $\widehat{V} = V \cup \{v_{ij} : \{i, j\} \in E\}$. Then $\widehat{G}$ has node set $\widehat{V}$ and its edge set is the union of all the cliques C_{ij} for $\{i, j\} \in E$. As an illustration Figure 7.2 shows the graph $\widehat{C_5}$.

Given $x \in \mathbb{Q}^E$, the construction of the new vector $\widehat{x} \in \mathbb{Q}^{\widehat{E}}$ is as follows: For each edge $\{i, j\} \in E$,

$$\widehat{x}_{ij} = x_{ij}, \tag{7.13}$$

$$\widehat{x}_{i,v_{ij}} = 4/5,\ \widehat{x}_{j,v_{ij}} = 3/5 \qquad \text{if } x_{ij} = 0, \tag{7.14}$$

$$\widehat{x}_{i,v_{ij}} = x_{ij},\ \widehat{x}_{j,v_{ij}} = 2x_{ij}^2 - 1 \quad \text{if } x_{ij} \neq 0. \tag{7.15}$$

We can now show that our construction for $\widehat{x}$ satisfies the two properties (7.10) and (7.11).

7.2.1 Lemma. *Given a graph $G = (V, E)$ and $x \in \mathbb{Q}^E$, let $\widehat{G} = (\widehat{V}, \widehat{E})$ be defined as above and let $\widehat{x} \in \mathbb{Q}^{\widehat{E}}$ be defined by (7.13)-(7.15). For fixed $k \geq 2$ we have that $x \in \mathscr{E}_k(G)$ if and only if $\widehat{x} \in \mathscr{E}_k(\widehat{G})$ and $\widehat{x} \in \text{ext } \mathscr{E}(\widehat{G})$.*

Proof. Sufficiency is clear so it remains to prove necessity. Applying Theorem 3.2.8, we find that the two matrices

$$\begin{pmatrix} 1 & 0 & 3/5 \\ 0 & 1 & 4/5 \\ 3/5 & 4/5 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_{ij} & x_{ij} \\ x_{ij} & 1 & 2x_{ij}^2 - 1 \\ x_{ij} & 2x_{ij}^2 - 1 & 1 \end{pmatrix} \text{ where } x_{ij} \in [-1, 1] \setminus \{0\}, \tag{7.16}$$

are extreme points of $\mathscr{E}_3$. Indeed, for the second matrix, if $|x_{ij}| = 1$ then it has rank one and if $|x_{ij}| < 1$ then it is an extreme point of rank 2. Therefore, for each edge $\{i, j\} \in E$, the restriction $\widehat{x}_{C_{ij}}$ of $\widehat{x}$ to the clique C_{ij} is an extreme point of $\mathscr{E}(C_{ij})$. By construction, $\widehat{G}$ is obtained as the clique sum of G with the cliques C_{ij}. As both matrices in (7.16) have rank at most 2 and as $k \geq 2$, Lemma 2.3.11 implies that $\widehat{x} \in \mathscr{E}_k(\widehat{G})$.

Finally, we show that $\widehat{x}$ is an extreme point of $\mathscr{E}(\widehat{G})$. Assume that $\widehat{x} = \sum_{k=1}^{m} \lambda_k \widehat{x}_k$ where $\lambda_k > 0$, $\sum_{k=1}^{m} \lambda_k = 1$ and $\widehat{x}_k \in \mathscr{E}(\widehat{G})$. For any $\{i, j\} \in E$, taking the projection onto the clique C_{ij} and using the fact that $\widehat{x}_{C_{ij}} \in \operatorname{ext} \mathscr{E}(C_{ij})$ we deduce that $(\widehat{x}_k)_{C_{ij}} = \widehat{x}_{C_{ij}}$ for all $k \in [m]$. As the cliques $\{C_{ij} : \{i, j\} \in E\}$ cover the graph $\widehat{G}$ it follows that $\widehat{x} = \widehat{x}_k$ for all $k \in [m]$. □

Combining these results we arrive at the main result of this section.

7.2.2 Theorem. *For any fixed $k \geq 2$, given a graph $G = (V, E)$ and rational edge weights $x \in \mathbb{Q}^E$, it is* NP*-hard to decide whether $x \in \operatorname{conv} \mathscr{E}_k(G)$.*

Proof. We show that the problem is hard already when the input is restricted to extreme points of $\mathscr{E}_k(G)$. By relation (7.9), for such points, testing membership in $\operatorname{conv} \mathscr{E}_k(G)$ is equivalent to testing membership in $\mathscr{E}_k(G)$.

In Theorems 7.1.5 and 7.1.10 we established that for any fixed $k \geq 2$ testing membership in $\mathscr{E}_k(G)$ is NP-hard. Using Lemma 7.2.1, testing membership in $\mathscr{E}_k(G)$ reduces to testing membership in $\operatorname{conv}\mathscr{E}_k(\widehat{G})$ for extreme points of $\mathscr{E}_k(\widehat{G})$. As the reduction described in Lemma 7.2.1 can be carried out in polynomial time, the latter problem is NP-hard. □

8

Grothendieck-type inequalities

Grothendieck's inequality is a tool of fundamental importance to many mathematical disciplines. The study of this inequality, its implications and its various generalizations has been, and continues to be, a highly active area of research. Grothendieck's inequality is relevant to a number of disparate areas ranging from approximation algorithms to communication complexity [8, 9, 29, 48, 105, 131]. Our goal in this chapter is to briefly introduce and give some background concerning the classical Grothendieck inequality. Moreover, we also introduce two variants of the classical Grothendieck inequality that will be studied in later chapters. For an extensive treatment of this vast topic the reader is referred to the extensive surveys of Pisier [104] and Khot and Naor [67].

8.1 Grothendieck's inequality

In this section we present the classical Grothendieck inequality. The formulation we will present here is due to to Lindenstauss and Pelczynski [85]. The inequality was initially proven by A. Grothendieck in [57], in a different but equivalent form.

8.1.1 Theorem. *[57] There exists a universal constant $K > 0$ such that for every matrix $A = (a_{ij}) \in \mathbb{R}^{n\times m}$ and every choice of unit vectors $u_1, \ldots, u_n, v_1, \ldots, v_m \in \mathbb{S}^{n+m-1}$ there exist signs $x_1, \ldots, x_n, y_1, \ldots, y_m \in \{\pm 1\}$ such that*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}\langle u_i, v_j\rangle \le K \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} x_i y_j.$$

Grothendieck's inequality admits a natural algorithmic interpretation as we will now see. Consider the quadratic integer program

$$\max_{x\in\{\pm1\}^n, y\in\{\pm1\}^m} \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} x_i y_j, \tag{8.1}$$

and its canonical semidefinite relaxation given by

$$\max_{\{u_i\}_{i=1}^n,\{v_j\}_{j=1}^m\subseteq\mathbb{S}^{n+m-1}} \sum_{i=1}^n\sum_{j=1}^m a_{ij}\langle u_i, v_j\rangle. \tag{8.2}$$

An equivalent way of expressing Theorem 8.1.1 is the following: there exists a universal constant $K > 0$ such that for every matrix $A = (a_{ij}) \in \mathbb{R}^{n\times m}$

$$\max_{\{u_i\}_{i=1}^n,\{v_j\}_{j=1}^m\subseteq\mathbb{S}^{n+m-1}} \sum_{i=1}^n\sum_{j=1}^m a_{ij}\langle u_i, v_j\rangle \le K \max_{x\in\{\pm1\}^n, y\in\{\pm1\}^m} \sum_{i=1}^n\sum_{j=1}^m a_{ij}x_i y_j. \tag{8.3}$$

In other words, the value of the program (8.1) can be approximated withing a constant factor by the value of the semidefinite program (8.2). Furthermore, it was shown in [9], that the proof of inequality (8.3) can be converted into an efficient rounding algorithm. In other words, given an optimal solution for (8.2) we can calculate signs $\{x_i\}_{i=1}^n, \{y_j\}_{j=1}^m$ whose value is within a constant factor from the value of (8.1).

The infimum over all $K > 0$ for which (8.3) holds is known as the *Grothendieck constant* and is denoted by K_G. Calculating the exact value of K_G is a long standing open problem, posed by Grothendieck himself. Nevertheless, it is known that

$$1.676... \le K_G < \frac{\pi}{2\log(1+\sqrt{2})}.$$

Here, the lower bound is due to Davie [38] and Reeds [113]. The upper bound is due to Krivine [72] and it was the best known bound for over thirty years. A major breakthrough took place in 2012, when it was shown that Krivine's upper bound is strict [29].

8.2 Generalizations of Grothendieck's inequality

The classical Grothendieck inequality has been generalized in many different directions in the literature. A non-exhaustive list of examples includes: the Grothendieck inequality for graphs [8], the positive semidefinite Grothendieck inequality [66], the non commutative Grothendieck inequality [103] and lastly, the rank-constrained Grothendieck inequality [31]. In this section we focus on two of these generalizations whose study forms the main body of the next two chapters.

8.2.1 The Grothendieck constant of a graph

Given a graph $G = ([n], E)$ and a vector of edge-weights $w = (w_{ij}) \in \mathbb{R}^E$, consider the following quadratic integer program over the hypercube:

$$\mathrm{ip}(G, w) = \max \sum_{ij\in E} w_{ij}x_i x_j \text{ s.t. } x_1, \dots, x_n \in \{\pm1\}, \tag{8.4}$$

and its canonical semidefinite programming relaxation

$$\mathrm{sdp}(G, w) = \max \sum_{ij\in E} w_{ij}\langle u_i, u_j\rangle \text{ s.t. } u_1, \dots, u_n \in \mathbb{S}^{n-1}. \tag{8.5}$$

8.2.1 Definition. *[8] The* Grothendieck constant *of a graph G, denoted by $\kappa(G)$, is defined as*

$$\kappa(G) = \sup_{w \in \mathbb{R}^E} \frac{\mathrm{sdp}(G,w)}{\mathrm{ip}(G,w)}. \tag{8.6}$$

In other words, $\kappa(G)$ is the integrality gap of relaxation (8.5) and thus it is equal to the smallest constant $K > 0$ such that, for every $w \in \mathbb{R}^E$,

$$\mathrm{sdp}(G,w) \leq K \cdot \mathrm{ip}(G,w). \tag{8.7}$$

Notice that the classical Grothendieck inequality (8.3) is a special case of (8.7) where G is the the complete bipartite graph $K_{n,m}$. This implies that

$$K_G = \sup_{n,m \in \mathbb{N}} \kappa(K_{n,m}).$$

As one might expect, the Grothendieck constant depends on the structure of the graph, as illustrated in the following theorem.

8.2.2 Theorem. *[8] For any graph G we have that*

$$\Omega(\log \omega(G)) \leq \kappa(G) \leq O(\log \vartheta(\overline{G})). \tag{8.8}$$

Here $\omega(G)$ denotes the maximum size of a clique in G and $\vartheta(\overline{G})$ the Lovász theta function of the complementary graph $\bar{G}$, for which it is known that $\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$ [87]. Hence, for the complete graph it follows that $\kappa(K_n) = \Theta(\log n)$.

Recently Briët et al. [31] showed that

$$\kappa(G) \leq \frac{2}{\pi \arcsin^{-1}(\vartheta(\overline{G}) - 1)}.$$

This bound gives Krivine's upper bound when specialized to bipartite graphs and improves the upper bound from (8.8) when $\vartheta(\overline{G})$ is small.

8.2.2 Higher rank Grothendieck inequalities

In this section we consider another family of relaxations for the quadratic integer program (8.4). Specifically, for any integer $r \geq 2$, consider the program

$$\mathrm{sdp}_r(G,w) = \max \sum_{ij \in E} w_{ij} \langle u_i, u_j \rangle \ \text{ s.t. } \ u_1, \ldots, u_n \in \mathbb{S}^{r-1}. \tag{8.9}$$

Observe that in (8.9) the vectors are restricted to lie in $\mathbb{S}^{r-1}$ and thus their corresponding Gram matrix will have rank at most r. This shows that (8.9) is a semidefinite program with a rank-r constraint.

As already explained in the introduction, the main motivation for studying program (8.9) comes from statistical mechanics and in particular from the *r-vector model*, introduced by Stanley [124]. As the rank function is non-convex and non-differentiable, such problems can be computationally challenging. This motivates the need to obtain tractable relaxations for (8.9).

8.2.3 Definition. *The* rank-r Grothendieck constant *of a graph G, denoted as $\kappa(r,G)$, is defined as*

$$\kappa(r,G) = \sup_{w \in \mathbb{R}^E} \frac{\mathrm{sdp}(G,w)}{\mathrm{sdp}_r(G,w)}. \tag{8.10}$$

In other words, $\kappa(r,G)$ is equal to the integrality gap of (8.5) when considered as a relaxation of the program (8.9). Notice that $\kappa(r,G)$ can be equivalently defined as smallest $K > 0$ such that for all $w \in \mathbb{R}^E$ the following inequality holds:

$$\mathrm{sdp}(G,w) \le K \cdot \mathrm{sdp}_r(G,w). \tag{8.11}$$

This quantity was introduced and studied in [31] the main motivation being the polynomial time approximation of ground states of spin glasses.

The upper bound from (8.8) has been generalized in this higher rank setting. Specifically, it was shown in [31] that for any graph G and any integer $r \in [1, \log \vartheta(\overline{G})]$ we have $\kappa(r,G) \le O\left(\log \vartheta(\overline{G})/r\right)$. So far there has been no progress in obtaining a lower bound, analogous to the one given in (8.8), for $\kappa(r,G)$.

9

The Grothendieck constant of some graph classes

In this chapter we investigate the Grothendieck constant for some specific graph classes. We prove some elementary properties of the Grothendieck constant of a graph and discuss the connections with the MAX CUT problem. The main result in this chapter is a closed-form expression for the Grothendieck constant of graphs with no K_5-minor. Additionally, we show that the integrality gap for clique-web inequalities, a wide class of valid inequalities for the cut polytope, is constant.

The content of this chapter is based on joint work with M. Laurent [81].

9.1 Introduction

Our goal in this chapter is to determine the Grothendieck constant for some specific graph classes. Our main result, is a closed-form expression for the Grothendieck constant of graphs with no K_5-minor.

Throughout this chapter we will use the following notation: For $w \in \mathbb{R}^E$, let $\kappa(G,w) = \mathrm{sdp}(G,w)/\mathrm{ip}(G,w)$ and $w(E) = \sum_{e\in E} w_e$. Our first observation is a geometric interpretation of $\kappa(G)$ as the smallest dilation of $\mathrm{CUT}^{\pm 1}(G)$ containing $\mathcal{E}(G)$.

9.1.1 Lemma. *For any graph G,*

$$\kappa(G) = \min\{K \ : \ \mathcal{E}(G) \subseteq K \cdot \mathrm{CUT}^{\pm 1}(G)\}.$$

Proof. Directly, since $\mathrm{ip}(G,w) = \max\limits_{x\in \mathrm{CUT}^{\pm 1}(G)} w^{\mathsf{T}} x$ and $\mathrm{sdp}(G,w) = \max\limits_{x\in\mathcal{E}(G)} w^{\mathsf{T}} x.$ □

As the origin lies in the interior of $\mathrm{CUT}^{\pm 1}(G)$, the polytope $\mathrm{CUT}^{\pm 1}(G)$ has a linear inequality description consisting of finitely many facet-defining inequalities of the form $w^T x \leq 1$. We now recall the switching operation (cf. Definition 4.1.4): Given $w \in \mathbb{R}^E$, its switching by $S \subseteq [n]$ is the vector $w^{\delta_G(S)} \in \mathbb{R}^E$ whose (i,j)-th entry is $-w_{ij}$ if the edge ij is cut by the partition $(S,[n]\setminus S)$ and w_{ij} otherwise.

Recall that the switching operation preserves valid inequalities and facet defining inequalities of the cut polytope (cf. Theorem 4.1.5).

We now observe that for any $S \subseteq [n]$ we have $\mathrm{sdp}(G,w) = \mathrm{sdp}(G, w^{\delta_G(S)})$ and $\mathrm{ip}(G,w) = \mathrm{ip}(G, w^{\delta_G(S)})$. The first claim follows by noting that if $x \in \mathscr{E}(G)$ then $x^{\delta_G(S)} \in \mathscr{E}(G)$; this is a direct consequence of Lemma 2.3.10. For the second claim notice that if $\delta_G(S') \in \mathrm{CUT}^{\pm 1}(G)$ then $\delta_G(S')^{\delta_G(S)} = \delta_G(S) \triangle \delta_G(S') \in \mathrm{CUT}^{\pm 1}(G)$ (since the symmetric difference of two cuts is again a cut).

This implies the next lemma which gives a useful reformulation for $\kappa(G)$.

9.1.2 Lemma. *For any graph G we have that*

$$\kappa(G) = \sup_{w \in \mathbb{R}^E} \kappa(G,w),$$

where the supremum ranges over all facet defining inequalities of $\mathrm{CUT}(G)$*, distinct up to switching.*

9.1.1 Connections with MAX CUT

Given $G = ([n], E)$ and $w \in \mathbb{R}^E$, recall that the MAX CUT problem asks for a cut of maximum weight cut in G. Thus we want to compute

$$\mathrm{mc}(G,w) = \max_{x \in \{\pm 1\}^n} \frac{1}{2} \sum_{ij \in E} w_{ij}(1 - x_i x_j) \tag{9.1}$$

Given a graph $G = ([n], E)$ and $w \in \mathbb{R}^E$ its *Laplacian matrix* is defined as

$$L_{G,w} = \sum_{ij \in E} w_{ij}(e_i - e_j)(e_i - e_j)^{\mathsf{T}}. \tag{9.2}$$

For any $x \in \mathbb{R}^n$ we have that $x^{\mathsf{T}} L_{G,w} x = \sum_{ij \in E} w_{ij}(x_i - x_j)^2$. This implies that for any $S \subseteq V$, $(\chi^{\delta_G(S)})^{\mathsf{T}} L_{G,w} \chi^{\delta_G(S)} = 4w(\delta(S))$ (recall we are working in ± 1 variables). This allows us to reformulate the MAX CUT problem as the follows:

$$\max_{x \in \{\pm 1\}^n} \frac{1}{4} x^{\mathsf{T}} L_{G,w} x = \max\{\frac{1}{4}\langle L_{G,w}, X\rangle : \mathrm{rank} X = 1,\ X \in \mathscr{E}_n\}. \tag{9.3}$$

Consider the canonical semidefinite programming relaxation of (9.3) obtained by relaxing the rank constraint:

$$\mathrm{sdp}_{\mathrm{GW}}(G,w) = \max_{X \in \mathscr{E}_n} \frac{1}{4}\langle L_{G,w}, X\rangle. \tag{9.4}$$

This semidefinite program was considered by Goemans and Williamson and yields the best known polynomial time approximation algorithm for MAX CUT.

Notice that $\mathrm{mc}(G,w) = \frac{1}{2}\big(w(E) + \mathrm{ip}(G,-w)\big)$, so the quadratic integer problem (8.4) and the MAX CUT problem (9.1) are affine transforms of each other. The same holds for their semidefinite relaxations (8.5) and (9.4), namely, $\mathrm{sdp}_{\mathrm{GW}}(G,w) = \frac{1}{2}\big(w(E) + \mathrm{sdp}(G,-w)\big)$. In particular, this implies that, given $w \in \mathbb{Q}^E$, deciding whether $\mathrm{ip}(G,w) = \mathrm{sdp}(G,w)$ is an NP-complete problem [106].

We continue with a useful lemma.

9.1.3 Lemma. *Consider a matrix* $A \in \mathcal{S}_+^n$ *and define* $B = \begin{pmatrix} 0 & A/2 \\ A/2 & 0 \end{pmatrix}$. *Then,*

$$\max_{Z \in \mathcal{E}_{2n}} \langle B, Z \rangle = \max_{X \in \mathcal{E}_n} \langle A, X \rangle \quad \text{and} \quad \max_{z \in \{\pm 1\}^{2n}} z^{\mathsf{T}} B z = \max_{x \in \{\pm 1\}^n} x^{\mathsf{T}} A x.$$

Proof. We only show the first equality, the proof of the second one being identical. Using the Gram decompositions of the feasible matrices and the form of B the first equality can be reformulated as:

$$\max_{u_i, v_j \in \mathbb{S}^{2n-1}} \sum_{i,j=1}^{n} A_{ij} \langle u_i, v_j \rangle = \max_{u_i \in \mathbb{S}^{n-1}} \sum_{i,j=1}^{n} A_{ij} \langle u_i, u_j \rangle. \tag{9.5}$$

Clearly $\max_{u_i, v_j \in \mathbb{S}^{2n-1}} \sum_{i,j=1}^{n} A_{ij} \langle u_i, v_j \rangle \geq \max_{u_i \in \mathbb{S}^{n-1}} \sum_{i,j=1}^{n} A_{ij} \langle u_i, u_j \rangle$ and for the other direction consider vectors $u_i^*, v_j^* \in \mathbb{S}^{2n-1}$ maximizing the left hand side of (9.5). By assumption the matrix A is psd so we can write it as $A = \sum_r a^r (a^r)^{\mathsf{T}}$. Then

$$\begin{aligned}
\sum_{i,j=1}^{n} A_{ij} \langle u_i^*, v_j^* \rangle = \sum_{i,j=1}^{n} \sum_r a_i^r a_j^r \langle u_i^*, v_j^* \rangle = \sum_r \sum_{i,j=1}^{n} a_i^r a_j^r \langle u_i^*, v_j^* \rangle = \\
\sum_r \langle \sum_{i=1}^{n} a_i^r u_i^*, \sum_{j=1}^{n} a_j^r v_j^* \rangle \leq \sum_r \| \sum_{i=1}^{n} a_i^r u_i^* \| \cdot \| \sum_{j=1}^{n} a^r v_j^* \| \leq \\
\sqrt{\sum_r \| \sum_{i=1}^{n} a_i^r u_i^* \|^2} \cdot \sqrt{\sum_r \| \sum_{j=1}^{n} a^r v_j^* \|^2} = \\
\sqrt{\sum_{i,j=1}^{n} A_{ij} \langle u_i^*, u_j^* \rangle} \cdot \sqrt{\sum_{i,j=1}^{n} A_{ij} \langle v_i^*, v_j^* \rangle} \leq \max_{u_i \in \mathbb{S}^{n-1}} \sum_{i,j=1}^{n} A_{ij} \langle u_i, u_j \rangle,
\end{aligned}$$

where the two inequalities follow by applying the Cauchy-Schwarz inequality. □

For $w \geq 0$ we have that $L_{G,w} \succeq 0$ (recall (9.2)) and thus Lemma 9.1.3 implies that $\mathrm{sdp}_{\mathrm{GW}}(G, w) = \max_{Z \in \mathcal{E}_{2n}} \langle B, Z \rangle$, where B has the form as in the lemma with $A/2 = L_{G,w}/8$. By the definition of the Grothendieck constant K_G, this implies that $\mathrm{sdp}_{\mathrm{GW}}(G, w) \leq K_G \cdot \mathrm{mc}(G, w)$. However, this approximation guarantee is not interesting since we know by [51] that $\mathrm{sdp}_{\mathrm{GW}}(G, w) \leq 1.138 \cdot \mathrm{mc}(G, w)$, while $K_G \geq 1.6$.

On the other hand, the Grothendieck constant $\kappa(G)$ bounds the semidefinite approximation for MAX CUT for edge weights satisfying $w(E) \geq 0$.

9.1.4 Lemma. *Let* $G = (V, E)$ *be a graph and let* $w \in \mathbb{R}^E$ *with* $w(E) \geq 0$. *Then,*

$$\mathrm{sdp}_{\mathrm{GW}}(G, w) \leq \kappa(G) \cdot \mathrm{mc}(G, w).$$

Proof. Indeed, $\mathrm{sdp}(G, -w) \leq \kappa(G) \cdot \mathrm{ip}(G, -w)$ and $w(E) \leq \kappa(G) \cdot w(E)$ imply

$$\frac{\mathrm{sdp}_{\mathrm{GW}}(G, w)}{\mathrm{mc}(G, w)} = \frac{w(E) + \mathrm{sdp}(G, -w)}{w(E) + \mathrm{ip}(G, -w)} \leq \kappa(G).$$

□

9.1.2 Behaviour under graph operations

In this section we investigate the behavior of the parameter $\kappa(\cdot)$ with respect to some basic graph operations. It follows immediately from the definition that $\kappa(\cdot)$ is monotone nonincreasing with respect to deleting edges. That is,

9.1.5 Lemma. *If $H \subseteq G$ then $\kappa(H) \leq \kappa(G)$.*

This is not true for the operation of contracting an edge. For instance, $\kappa(K_2) = 1 < \kappa(C_3) = 3/2$, while $\kappa(C_4) < \kappa(C_3) = 3/2$ (cf. Theorem 9.2.1). So $\kappa(G)$ and $\kappa(G/e)$ are not comparable in general.

We now show that Grothendieck constant behaves nicely with respect to the clique sum operation.

9.1.6 Lemma. *Assume G is the clique k-sum ($k \leq 3$) of G_1 and G_2. Then,*

$$\kappa(G) = \max(\kappa(G_1), \kappa(G_2)).$$

Proof. Let $\lambda = \max(\kappa(G_1), \kappa(G_2))$ and $n = |V|$. The inequality $\kappa(G) \geq \lambda$ follows from Lemma 9.1.5. For the other direction, let $x \in \mathscr{E}(G)$ and $X \in \mathscr{E}_n$ such that $x = \pi_E(X)$; we have to show that $x \in \lambda \cdot \mathrm{CUT}(G)$. Let X_i denote the principal submatrix of X indexed by V_i, for $i = 1, 2$. As $\mathscr{E}(G_i) \subseteq \kappa(G_i) \cdot \mathrm{CUT}(G_i) \subseteq \lambda \cdot \mathrm{CUT}(G_i)$, we deduce that $\pi_{E_i}(X_i) \in \lambda \cdot \mathrm{CUT}(G_i)$. Using Lemma 4.2.6 the claim follows. □

9.2 Computing the Grothendieck constant

In this section we establish our main results, namely the closed form formulas for the Grothendieck constant of circuits and graphs with no K_5-minor.

9.2.1 The case of circuits

Using the parametrizations of $\mathrm{MET}^{\pm 1}(C_n)$ and $\mathscr{E}(C_n)$ given by Theorems 4.2.3 and 4.3.4, respectively, we are able to compute the Grothendieck constant of the circuits. Specifically,

9.2.1 Theorem. *The Grothendieck constant of a circuit C_n of length $n \geq 3$ is equal to*

$$\kappa(C_n) = \frac{n}{n-2} \cos\left(\frac{\pi}{n}\right).$$

Proof. By Lemma 9.1.2 it suffices to compute $\kappa(C_n, w)$ for facet defining inequalities of $\mathrm{CUT}^{\pm 1}(C_n)$. By Theorem 4.2.3 we have that $\mathrm{CUT}^{\pm 1}(C_n) = \mathrm{MET}^{\pm 1}(C_n)$ and by Theorem 4.2.4 we know that the facets of $\mathrm{MET}^{\pm 1}(C_n)$ correspond exactly to the circuit inequalities given in (4.5). It is easy to see that all circuit inequalities are switching equivalent (cf. Lemma 4.2.5) and thus it suffices to consider one of them; For instance, we can choose $-x(C_n) \leq n-2$ and $x_e - x(E \setminus \{e\}) \leq n-2$ for even n.

For n odd we have that $\kappa(C_n) = \kappa(C_n, -e/(n-2)) = \kappa(C_n, -e)$. Since the inequality $-x(C_n) \leq n-2$ defines a facet of $\mathrm{CUT}^{\pm 1}(G)$ it follows that $\mathrm{ip}(C_n, -e) = n-2$. Thus it now suffices to show that $\mathrm{sdp}(C_n, -e) = n\cos(\pi/n)$ as this will give the desired value for $\kappa(C_n)$. For n odd, it is known that $\mathrm{sdp}_{\mathrm{GW}}(C_n, e) = \frac{n}{4}\left(2 + 2\cos\frac{\pi}{n}\right)$ (see [107]), which implies that $\mathrm{sdp}(C_n, -e) = 2\,\mathrm{sdp}_{\mathrm{GW}}(C_n, e) - n = n\cos(\pi/n)$. Alternatively, this can also be easily verified using the parametrization of $\mathscr{E}(C_n)$ from Theorem 4.3.4.

One can also compute $\mathrm{sdp}(C_n, w)$ for n even and $w = (-1, 1, \ldots, 1)$ using Theorem 4.3.4; it turns out that this has also been computed in [131] in the context of quantum information theory. □

9.2.2 The case of K_5-minor free graphs

Since K_5-minor free graphs are 4-colorable [130], we deduce from (8.8) that their Grothendieck constant $\kappa(G)$ is bounded. In this section we give a closed-form formula for $\kappa(G)$ in terms of the girth of G.

9.2.2 Theorem. *If G is a graph with no K_5 minor (and G is not a forest), then*

$$\kappa(G) = \frac{g}{g-2} \cos\left(\frac{\pi}{g}\right),$$

where g is the minimum length of a circuit in G.

Proof. Directly from Theorem 9.2.1 using the facts that all facets of G are supported by circuits (Theorem 4.2.3) and that the function $\frac{n}{n-2}\cos(\frac{\pi}{n})$ is monotone nonincreasing in n. □

As a direct application, we recover the values $\kappa(K_{2,n}) = \kappa(K_{3,n}) = \sqrt{2}$, for $n \geq 3$. Using the descriptions of $\mathrm{CUT}(K_{2,n})$ and $\mathrm{CUT}(K_{3,n})$ ($n \geq 3$), these values were also computed in the context of quantum information theory [48].

9.2.3 Graphs whose cut polytope is defined by inequalities supported by at most k points

In this section we show that the Grothendieck constant can be bounded in terms of the size of the supports of the inequalities defining facets of the cut polytope. The *support graph* of an inequality $w^{\mathsf{T}}x \leq 1$ is the graph $H = (W, F)$, where $F = \{ij \in E : w_{ij} \neq 0\}$ and W is the set of nodes covered by F. We say that $w^{\mathsf{T}}x \leq 1$ is *supported by at most k points* when $|W| \leq k$. For instance, a triangle inequality is supported by 3 points.

Fix an integer $k \geq 2$. Let $\mathscr{R}_k(K_n) \subseteq \mathbb{R}^{\binom{n}{2}}$ be the polyhedron defined by all valid inequalities for $\mathrm{CUT}_n^{\pm 1}$ supported by at most k points. For a graph $G = ([n], E)$, set

$$\mathscr{R}_k(G) = \pi_E(\mathscr{R}_k(K_n)).$$

Notice that the convex sets $\mathscr{R}_k(G)$ ($k \in [n]$) form a hierarchy of relaxations for $\mathrm{CUT}^{\pm 1}(G)$, namely

$$\mathrm{CUT}^{\pm 1}(G) = \mathscr{R}_n(G) \subseteq \mathscr{R}_{n-1}(G) \subseteq \ldots \subseteq \mathscr{R}_1(G).$$

For instance, $\mathscr{R}_3(K_n) = \mathrm{MET}^{\pm 1}(K_n)$, and thus $\mathscr{R}_3(G) = \mathrm{MET}^{\pm 1}(G)$.

For the remainder of this section let $\mathscr{B}_k$ denote the class of all graphs G for which $\mathrm{CUT}^{\pm 1}(G) = \mathscr{R}_k(G)$. For instance, $\mathscr{B}_2$ consists of all forests (thus the K_3-minor free graphs) and $\mathscr{B}_3$ of the K_5-minor free graphs. The next result shows that such a characterization exists for any integer $k \geq 1$.

9.2.3 Theorem. *The class $\mathscr{B}_k$ is closed under taking minors for any integer $k \geq 1$.*

Proof. Let $G = ([n], E)$ be a graph with $G \in \mathscr{B}_k$. We first show that $G \setminus e \in \mathscr{B}_k$ for any $e \in E$. By definition we have to show that $\mathscr{R}_k(G \setminus e) \subseteq \mathrm{CUT}^{\pm 1}(G)$. Let $x \in \mathscr{R}_k(G \setminus e)$. Then $x = \pi_{G\setminus e}(y)$ where $y \in \mathscr{R}_k(K_n)$. Then $(x, y_e) \in \mathscr{R}_k(G)$ and thus by hypothesis $(x, y_e) \in \mathrm{CUT}^{\pm 1}(G)$. This implies that $x \in \mathrm{CUT}^{\pm 1}(G \setminus e)$.

It remains to verify that $\mathscr{B}_k$ is closed under edge contraction. Let $G' = G/e = (V', E')$, where $e = (1, 2)$ and $V^{'} = \{2, \ldots, n\}$. Given $y \in \mathbb{R}^{E'}$, define its extension $\tilde{y} \in \mathbb{R}^E$ by $\tilde{y}_{12} = 1$, $\tilde{y}_{1i} = y_{2i}$ if $1i \in E$ with $i \geq 3$, $\tilde{y}_{2i} = y_{2i}$ if $2i \in E$ with $i \geq 3$, and $\tilde{y}_{ij} = y_{ij}$ if $ij \in E$ with $i, j \geq 3$. We now show that $\tilde{y} \in \mathrm{CUT}(G)$ if and only if $y \in \mathrm{CUT}(G/e)$. Assume that $\tilde{y} \in \mathrm{CUT}(G)$ and say $\tilde{y} = \sum_S \lambda_S \chi^{\delta_G(S)}$ where $\lambda_s \geq 0$ and $\sum_S \lambda_S = 1$. As $\tilde{y}_e = 1$ it follows that $\chi_e^{\delta_G(S)} = 1$ for all S and thus all the cuts defined by the sets S miss the edge $e = 12$. Thus we can assume wlog that $\{1, 2\} \subseteq S$ for all S. Then it is easy to see that $y = \sum_S \lambda_S \chi^{\delta_{G'}(S \setminus 1)}$. The other direction follows similarly.

We can now conclude the proof. Assuming that $G \in \mathscr{B}_k$ we want to show that $G' = G/e \in \mathscr{B}_k$, i.e., $\mathscr{R}_k(G/e) \subseteq \mathrm{CUT}^{\pm 1}(G/e)$. For this let $z \in \mathscr{R}_k(G/e)$ and say that $z = \pi_{E'}(y)$ where $y = (y_{ij})_{2 \leq i < j \leq n} \in \mathscr{R}_k(K_{n-1})$. The next step is to extend the vector y to a vector $\hat{y} \in \mathbb{R}^{\binom{n}{2}}$ which is defined as follows:

$$\hat{y}_{12} = 1,\ \hat{y}_{1i} = y_{2i} \text{ for } i \geq 3,\ \hat{y}_{2i} = y_{2i} \text{ for } i \geq 3 \text{ and } \hat{y}_{ij} = y_{ij} \text{ for } 3 \leq i < j \leq n.$$

Our next goal is to show that $\hat{y} \in \mathscr{R}_k(K_n)$. For this let $w^\mathsf{T} x \leq 1$ be a valid inequality for CUT_n supported by at most k points. We need to show that $w^\mathsf{T} \hat{y} \leq 1$. Define the new inequality on $x = (x_{ij})_{2 \leq i < j \leq n}$:

$$b^\mathsf{T} x = \sum_{i=3}^{n} (w_{1i} + w_{2i}) x_{2i} + \sum_{3 \leq i < j \leq n} w_{ij} x_{ij} \leq 1 - w_{12}.$$

Obviously, this inequality is supported by at most k points. Our next goal is to show that it is valid for CUT_{n-1} and thus it is one of the defining inequalities of $\mathscr{R}_k(K_{n-1})$. We need to show that for every $S \subseteq \{2, \ldots, n\}$ we have that $b^\mathsf{T} \chi^{\delta_{K_{n-1}}(S)} \leq 1$. For this consider a cut of K_{n-1} defined by $S \subseteq \{2, \ldots, n\}$ and wlog assume that $2 \in S$. Notice that the cut $\delta_{K_n}(S \cup 1)$ is a cut of K_n that extends the cut $\delta_{K_{n-1}}(S)$. Then, it follows by assumption that $w^\mathsf{T} \chi^{\delta_{K_n}(S \cup 1)} \leq 1$. Notice that in the cut $\delta_{K_n}(S \cup 1)$ the edge $\{1, 2\}$ is not cut, i.e., $\chi_{12}^{\delta_{K_n}(S \cup 1)} = 1$. Moreover, notice that $\chi_{1i}^{\delta_{K_n}(S \cup 1)} = \chi_{2i}^{\delta_{K_{n-1}}(S)}$ for $i \geq 3$, $\chi_{2i}^{\delta_{K_n}(S \cup 1)} = \chi_{2i}^{\delta_{K_{n-1}}(S)}$ for $i \geq 3$ and $\chi_{ij}^{\delta_{K_n}(S \cup 1)} = \chi_{ij}^{\delta_{K_{n-1}}(S)}$ for $3 \leq i < j \leq n$. Substituting all these relations into $w^\mathsf{T} \chi^{\delta_{K_n}(S \cup 1)} \leq 1$ the claim follows.

Summarizing we just showed that the inequality $b^\mathsf{T} x \leq 1 - w_{12}$ is a defining inequality of $\mathscr{R}_k(K_{n-1})$. As $y \in \mathscr{R}_k(K_{n-1})$ we have that $b^\mathsf{T} y \leq 1 - w_{12}$ and a simple calculation shows that this is equivalent to $w^\mathsf{T} \hat{y} \leq 1$. This implies that $\hat{y} \in \mathscr{R}_k(K_n)$.

To conclude the proof notice that the fact that $\hat{y} \in \mathscr{R}_k(K_n)$ combined with the assumption that $G \in \mathscr{B}_k$ imply that $\pi_E(\hat{y}) \in \mathrm{CUT}^{\pm 1}(G)$ which by the discussion in the second paragraph shows that $z \in \mathrm{CUT}^{\pm 1}(G/e)$. $\square$

Notice that for $G \in \mathscr{B}_2$, $\kappa(G) = \kappa(K_2) = 1$. Moreover, Theorem 9.2.2 implies that $\kappa(G) \leq \kappa(K_3) = 3/2$ for $G \in \mathscr{B}_3$. The next theorem shows that this pattern extends to any integer $k \geq 1$.

9.2.4 Theorem. *If $G \in \mathscr{B}_k$ then $\kappa(G) \leq \kappa(K_k)$ and this bound is tight.*

Proof. Consider a graph $G \in \mathscr{B}_k$. Since $\mathscr{R}_K(G) = \mathrm{CUT}^{\pm 1}(G)$ it is enough to show that $\mathscr{E}(G) \subseteq \kappa(K_k) \cdot \mathscr{R}_k(G)$. Moreover, it suffices to consider only $G = K_n$, as the general result follows by taking projections. Let $y \in \mathscr{E}(K_n)$ and let $w^T x \le 1$ be a valid inequality for CUT_n with support $H = (W, F)$ where $|W| \le k$. Then, $w^T y = \pi_F(w)^T \pi_F(y) \le \mathrm{sdp}(H, \pi_F(w)) \le \kappa(H) \cdot \mathrm{ip}(H, \pi_F(w)) \le \kappa(K_k)$, where we use the facts that $\kappa(H) \le \kappa(K_k)$ and $\mathrm{ip}(H, \pi_F(w)) \le 1$ for the right most inequality. Lastly notice that this bound is tight since $K_k \in \mathscr{B}_k$. □

One can verify that $\kappa(K_7) = 3/2$ (see [78]). Hence, $\kappa(G) \le 3/2$ for all $G \in \mathscr{B}_7$.

9.3 Integrality gap of clique-web inequalities

As we have already seen, Theorem 8.2.2 implies that $\kappa(K_n) = \Theta(\log n)$. In view of this, it is interesting to identify inequalities that achieve this integrality gap. This was posed as an open question in [8] and instances with large integrality gap are given in [13]. In this section we show that the integrality gap is constant for the clique-web inequalities, a wide class of valid inequalities for $\mathrm{CUT}(K_n)$.

Throughout this section we will be using the following shorthand notation: For a vector $x \in \mathbb{R}^{\binom{n}{2}}$ and a graph $G = ([n], E)$ we will denote $x(G) = \sum_{ij \in E} x_{ij}$. Given integers p and r with $p \ge 2r + 3$, the *antiweb graph* AW_p^r is the graph with vertex set $[p]$, and with edges $(i, i+1), \ldots, (i, i+r)$ for $i \in [p]$, where the indices are taken modulo p. The *web graph* W_p^r is defined as the complement of AW_p^r in K_p. Call the set of edges $(i, i+s)$ for $i \in [p]$ (indices taken modulo p) the *s-th band*, so that AW_p^r consists of the first r bands in K_p.

Let p, q, r, n be integers satisfying $p - q = 2r + 1$, $q \ge 2, n = p + q$. The (pure) *clique-web inequality* with parameters n, p, q, r is the inequality

$$-x(K_q) - \sum_{\substack{1 \le i \le q \\ q+1 \le j \le n}} x_{ij} - x(\mathrm{W}_p^r) \le q(r+1). \tag{9.6}$$

The support graph of (9.6), denoted by CW_p^r, consists of a clique on the first q nodes, a web on the last p ones, and a complete bipartite graph between them. It is known that pure clique-web inequalities define facets of $\mathrm{CUT}_n^{\pm 1}$ (this is in general not true in the nonpure case) [44, Section 29.4]. Note that hypermetric and bicycle odd wheel inequalities arise as special cases of (9.6), for $r = 0$ and $r = \frac{n-5}{2}$, respectively (see [44]).

Since $x(W_p^r) = x(K_p) - x(AW_p^r)$ it follows that the left-hand side of (9.6) is equal to $-x(K_{p+q}) + x(\mathrm{AW}_p^r)$. Using this we will now show that

$$\mathrm{sdp}(\mathrm{CW}_p^r, -e) \le q(r+1) + (2r+1)^2/2. \tag{9.7}$$

Clearly, $\mathrm{sdp}(\mathrm{CW}_p^r, -e) \le \max\{-x(K_{p+q}) : X \in \mathscr{E}_n\} + \max\{x(AW_p^r) : X \in \mathscr{E}_p\}$ and we now proceed to bound each of these terms separately. The first term is equal to $\max\{-\sum_{1 \le i < j \le n} X_{ij} : X \in \mathscr{E}_n\}$ which after symmetry reduction can be seen to be equal with $\max\{-\binom{n}{2} a : \frac{-1}{n-1} \le a \le 1\} = \frac{n}{2}$. The second term is equal to $\max\{\sum_{ij \in AW_p^r} X_{ij} : X \in \mathscr{E}_p\}$ which is upper bounded by rp since $X_{ij} \le 1$ for all $i, j \in [p]$ and the number of edges of AW_p^r is equal to rp. Summing up we get that $\mathrm{sdp}(\mathrm{CW}_p^r, -e)$ is upper bounded by $n/2 + pr = q(r+1) + (2r+1)^2/2$.

This directly implies the following:

9.3.1 Lemma. *The integrality gap of a clique-web inequality with $q \geq 2r+1$ is upper bounded by 2.*

Proof. From (9.7) we have that $\kappa(\mathrm{CW}_p^r, -e) \leq 1 + \frac{(2r+1)^2}{q(2r+2)}$ and the claim follows. □

We now consider the case when $q \leq 2r$. We note that this inequality implies

$$p \geq 2q, \tag{9.8}$$

a fact that we will use a number of times for the remainder of this section.

9.3.2 Theorem. *The integrality gap of a clique-web inequality with $q \leq 2r$ is upper bounded by 3.*

Proof. We can rewrite $\mathrm{sdp}(\mathrm{CW}_p^r, -e)$ as

$$\max_{X \in \mathscr{E}_n} - \sum_{ij \in K_q} X_{ij} - \sum_{\substack{1 \leq i \leq q \\ q+1 \leq j \leq n}} X_{ij} - \sum_{ij \in \mathrm{W}_p^r} X_{ij}. \tag{9.9}$$

Notice that the program (9.9) is invariant under the action of the full symmetric group S_q acting on the row/column indices in $[q]$. Moreover, (9.9) is invariant under the action of the group of cyclic permutations in S_p acting on the row/column indices in $\{q+1, .., n\}$. Then we can find an optimal solution X in the invariant subspace which has the form

$$X = \begin{pmatrix} aJ_{q,q} + (1-a)I_q & bJ_{q,p} \\ bJ_{p,q} & X^p \end{pmatrix},$$

where $X_{ij}^p = c_{|j-i \bmod p|}$ for $q+1 \leq i \neq j \leq n$ and $X_{ii}^p = 1$ for $q+1 \leq i \leq n$.

One can easily verify that $X \succeq 0$ if and only if $Y = \begin{pmatrix} \beta & be^{\mathsf{T}} \\ be & X_p \end{pmatrix} \succeq 0$, after setting $\beta = \frac{(q-1)a+1}{q}$.

Consider first the case when q is even; so p is odd, all bands in W_p^r have size p, and the objective function in (9.9) reads

$$\frac{q}{2}(1 - q\beta) - pqb - p(c_1 + \ldots + c_{q/2}). \tag{9.10}$$

If $\beta = 0$, as Y is psd it follows that $b = 0$ and then Lemma 4.3.6 implies that $c_s \geq -\cos(\pi/p)$ for all $1 \geq s \leq q/2$. Indeed each band of W_p^r is a circuit or a disjoint union of circuits (e.g. the first band of W_9^2 is a union of three triangles). As p is odd, at least one of these circuits is an odd circuit of size $p' \leq p$, so that Lemma 4.3.6 implies that the entries on the band are at least $-\cos(\pi/p') \geq -\cos(\pi/p)$. Setting $\gamma = \cos(\pi/p)$ we have that (9.10) becomes

$$\frac{q}{2} - p(c_1 + \ldots + c_{q/2}) \leq \frac{q}{2}(p\gamma + 1) \leq \frac{q}{2}(p+1) = \frac{q}{2}(q + 2r + 2) \leq 2\,q(r+1), \tag{9.11}$$

where the last inequality follows from $q \leq 2r$.

Assume now $\beta > 0$. Taking the Schur complement in Y with respect to the entry β, we can rewrite the condition $Y \succeq 0$ as $X^p - \frac{b^2}{\beta} J \succeq 0$. If $\beta = b^2$, we have that the matrix $X_p - J$ is positive semidefinite and all its diagonal elements are equal to zero.

This implies that $c_s = 1$ for all s and then (9.10) becomes $\frac{-q^2}{2}b^2 - pqb - \frac{pq}{2} + \frac{q}{2}$. This is a quadratic polynomial in b whose maximum in $[-1, 1]$ is equal to

$$q(r+1), \tag{9.12}$$

attained at $b = -1$ (its maximum in $\mathbb{R}$ is attained at $-p/q$ which is smaller than -1 by (9.8)).

Next assume $\beta > b^2$ and notice that $Y \succeq 0$ is equivalent to $Z = \frac{\beta}{\beta - b^2}X_p - \frac{b^2}{\beta - b^2}J \in \mathscr{E}_p$. As above, Lemma 4.3.6 permits to bound the entries of Z as follows: $\frac{\beta}{\beta - b^2}c_s - \frac{b^2}{\beta - b^2} \geq -\gamma$ for $1 \leq s \leq q/2$. Therefore, the program (9.9) is upper bounded by

$$\begin{array}{ll} \max\limits_{b,c,\beta} & \frac{q}{2}(1-q\beta) - pqb - cpq/2 \\ \text{s.t.} & \beta(c+\gamma) \geq b^2(\gamma+1) \\ & b^2 < \beta \leq 1,\ -1 \leq b, c \leq 1. \end{array} \tag{9.13}$$

At optimality, equality $\beta(c+\gamma) = b^2(\gamma+1)$ holds. This permits to express c in terms of b, β and to rewrite the objective function of (9.13) as

$$-\frac{pq}{2}\frac{(\gamma+1)}{\beta}b^2 - pqb + \frac{pq}{2}\gamma + \frac{q}{2}(1-q\beta). \tag{9.14}$$

For a fixed value of β, the maximum of this quadratic function in b is attained at $b = -\frac{\beta}{\gamma+1}$ and is equal to

$$\frac{q}{2}(1-q\beta) + \frac{pq}{2}\left(\frac{\beta}{\gamma+1} + \gamma\right) = \frac{q}{2}\left(\beta(\frac{p}{\gamma+1} - q) + p\gamma + 1\right). \tag{9.15}$$

By (9.8), it follows that $\frac{p}{\gamma+1} \geq q$ and viewing (9.15) as a linear function in β it is increasing and thus it is maximized when $\beta = 1$. Substituting in (9.15) we see that the maximum of (9.13) is equal to $\frac{pq}{2}(\gamma + \frac{1}{\gamma+1}) - \frac{q(q-1)}{2}$. Hence, using $q \leq 2r$ and $\gamma + \frac{1}{\gamma+1} \leq \frac{3}{2}$, we deduce that this maximum is upper bounded by

$$3q(r+1). \tag{9.16}$$

Combining (9.11), (9.12) and (9.16), this concludes the proof that the integrality gap of the clique-web inequality is at most 3 when q is even.

Consider now the case when q is odd. Then p is even and W_p^r consists of $(q-1)/2$ bands of size p and one band of size $p/2$. The treatment is analogous to the case q even, except we must replace the objective function in (9.10) by

$$\frac{q}{2}(1-q\beta) - pqb - p(c_1 + \ldots + c_{\frac{q-1}{2}}) - \frac{p}{2}c_{\frac{q+1}{2}} \tag{9.17}$$

and, as p is even, the values on the bands can only be claimed to lie in $[-1, 1]$ by Lemma 4.3.6 (which amounts to setting $\gamma = 1$ in the above argument). Specifically, if $\beta = 0$ we can upper bound the objective function (9.17) by $2q(r+1)$ and, if $\beta = b^2$, we can upper bound (9.17) by $q(r+1)$. Finally, if $\beta > b^2$, as above we do a Schur complement and obtain $\frac{\beta}{\beta - b^2}c_s - \frac{b^2}{\beta - b^2} \geq -1$, so that (9.17) is upper bounded by the program (9.13) setting there $\gamma = 1$. Hence the integrality gap of the clique-web inequality is also bounded by 3 for q odd. □

10

The extreme Gram dimension of a graph

Given a graph $G = ([n], E)$ and a vector $w \in \mathbb{R}^E$ consider the semidefinite program

$$\max \sum_{ij \in E} w_{ij} X_{ij} \ \text{ s.t. } \ X_{ii} = 1 \ (i \in [n]), \ X \succeq 0. \tag{P_G^w}$$

The optimal value of (P_G^w) is attained since its feasible region is a compact set. Our main goal in this chapter is to find ways to exploit the combinatorial structure of the graph G, in order to get guarantees for the existence of low-rank optimal solutions to (P_G^w). To study this problem we introduce a new graph parameter, called the extreme Gram dimension of a graph, which we denote by $\text{egd}(\cdot)$. For any graph G, $\text{egd}(G)$ is defined as the smallest integer $r \geq 1$ such that for every $w \in \mathbb{R}^E$, (P_G^w) has an optimal solution of rank at most r. Our first result in this chapter is to show that this parameter is minor monotone. Thus, for any fixed $r \geq 1$, the graphs satisfying $\text{egd}(\cdot) \leq r$ can be characterized by a list of minimal forbidden minors. For the case $r = 1$, the only excluded minor is the graph K_3 [74]. Our main result in this chapter is to identify the minimal forbidden minors for the case $r = 2$. Additionally, we introduce a new a treewidth-like graph parameter, denoted by $\text{la}_{\boxtimes}(\cdot)$, which we call the *strong largeur d'arborescence*. For a graph G, $\text{la}_{\boxtimes}(G)$ is defined as the smallest integer $k \geq 1$ such that G is a minor of the strong graph product $T \boxtimes K_k$, where T is a tree and K_k denotes the complete graph on k vertices. In this chapter we show that the extreme Gram dimension is upper bounded by $\text{la}_{\boxtimes}(\cdot)$. Lastly, we obtain the forbidden minor characterization of graphs with $\text{la}_{\boxtimes}(G) \leq 2$.

The content of this chapter is based on joint work with M. E.-Nagy and M. Laurent [46].

10.1 Introduction

Consider a graph $G = ([n], E)$ and a vector of edge weights $w \in \mathbb{R}^E$. In this chapter we focus on semidefinite programs of the form

$$\mathrm{sdp}(G, w) = \max \sum_{ij \in E} w_{ij} X_{ij} \;\text{ s.t. }\; X \in \mathscr{E}_n, \tag{P_G^w}$$

where $\mathscr{E}_n$ denotes the set of n-by-n correlation matrices (positive semidefinite matrices with diagonal entries equal to one). Our main objective in this chapter is to exploit the combinatorial structure of the graph G, in order to get guarantees for the existence of low-rank optimal solutions to (P_G^w).

Since the feasible region of (P_G^w) is a compact set and the objective function is linear it follows that for any $w \in \mathbb{R}^E$, its optimal value is attained. Then, Theorem 1.2.1 applied to (P_G^w) implies that for any $w \in \mathbb{R}^E$, the program (P_G^w) has an optimal solution of rank at most $\mathrm{gd}(G)$. Nevertheless, the bound from Theorem 1.2.1 is valid for arbitrary SDP's and does not take into account the specific structure of the problem at hand. Indeed, problem (P_G^w) has the property that its constraints impose conditions only on the diagonal entries and thus the only contribution to the aggregate sparsity pattern comes from the objective function.

As we will see in this chapter, for semidefinite programs of the form (P_G^w) we can improve on the $\mathrm{gd}(G)$ bound. In order to achieve this we introduce a new graph parameter, called the *extreme Gram dimension* of graph, which we denote by $\mathrm{egd}(\cdot)$. For a graph $G = ([n], E)$, $\mathrm{egd}(G)$ is defined as the smallest integer $r \geq 1$, such that for any $w \in \mathbb{R}^E$, the program (P_G^w) has an optimal solution of rank at most r. Notice that since the optimal value of (P_G^w) is attained, the extreme Gram dimension of a graph is well defined and upper bounded by the number of nodes.

Our first goal is to give a reformulation for the extreme Gram dimension which spells out the link of this parameter with the Gram dimension and explains why we chose to name the parameter in this way.

Recall that $\mathscr{E}_{n,r}$ denotes the set of n-by-n correlation matrices of rank at most r. Moreover, for a graph $G = ([n], E)$, π_E denotes the projection from $\mathscr{S}^n$ onto the subspace $\mathbb{R}^E$ indexed by the edge set of G (recall (4.9)) and $\mathscr{E}(G) = \pi_E(\mathscr{E}_n)$.

For a graph $G = ([n], E)$ and $w \in \mathbb{R}^E$, consider the rank-constrained SDP:

$$\mathrm{sdp}_r(G, w) = \max \sum_{ij \in E} w_{ij} X_{ij} \;\text{ s.t. }\; X \in \mathscr{E}_{n,r}, \tag{10.1}$$

Then, $\mathrm{egd}(G)$ can be reformulated as the smallest integer $r \geq 1$ for which

$$\mathrm{sdp}(G, w) = \mathrm{sdp}_r(G, w) \text{ for all } w \in \mathbb{R}^E. \tag{10.2}$$

Notice that program (P_G^w) corresponds to optimization over $\mathscr{E}(G)$ since it can be reformulated as

$$\mathrm{sdp}(G, w) = \max\; w^{\mathsf{T}} x \;\text{ s.t. }\; x \in \mathscr{E}(G).$$

Moreover, since the objective function of the program (10.1) is linear, it follows that program (10.1) corresponds to optimization over $\pi_E(\mathrm{conv}(\mathscr{E}_{n,r}))$, i.e.,

$$\mathrm{sdp}_r(G, w) = \max\; w^{\mathsf{T}} x \;\text{ s.t. }\; x \in \pi_E(\mathrm{conv}(\mathscr{E}_{n,r})).$$

Then, in view of (10.2) we get the following reformulation for the $\mathrm{egd}(\cdot)$.

10.1.1 Definition. *The* extreme Gram dimension *of a graph* $G = ([n], E)$ *is the smallest integer* $r \geq 1$ *for which*

$$\mathscr{E}(G) = \pi_E(\text{conv}(\mathscr{E}_{n,r})). \tag{10.3}$$

Notice that the inclusion $\pi_E(\text{conv}(\mathscr{E}_{n,r})) \subseteq \mathscr{E}(G)$ is always valid and thus the extreme Gram dimension of a graph is equal to the smallest $r \geq 1$ for which $\mathscr{E}(G) \subseteq \pi_E(\text{conv}(\mathscr{E}_{n,r}))$. Moreover, as we have already seen, $\mathscr{E}(G)$ is a compact and convex subset of $\mathbb{R}^E$ and thus, by the Krein–Milman theorem (see [24, Theorem 3.3]), $\mathscr{E}(G)$ is equal to the convex hull of its extreme points. Then, since $\pi_E(\text{conv}(\mathscr{E}_{n,r}))$ is a convex set, $\text{egd}(G)$ is equal to the smallest integer $r \geq 1$ for which $\text{ext}\,\mathscr{E}(G) \subseteq \pi_E(\text{conv}(\mathscr{E}_{n,r}))$. Lastly, notice that for a vector $x \in \text{ext}\,\mathscr{E}(G)$, we have $x \in \pi_E(\text{conv}(\mathscr{E}_{n,r}))$ if and only if $x \in \pi_E(\mathscr{E}_{n,r})$. This allows us to reformulate the $\text{egd}(\cdot)$ as the smallest $r \geq 1$ for which:

$$\text{ext}\,\mathscr{E}(G) \subseteq \pi_E(\mathscr{E}_{n,r}). \tag{10.4}$$

In other words, equation (10.4) says that the $\text{egd}(G)$ is equal to the smallest $r \geq 1$ for which every extreme point of $\mathscr{E}(G)$ has a psd completion of rank at most r.

Recall that for a vector $x \in \mathscr{E}(G)$, $\text{gd}(G, x)$ is defined as the smallest rank of a completion to a correlation matrix of the G-partial matrix defined by x. This allows us to give yet another reformulation for the extreme Gram dimension.

10.1.2 Lemma. *For any graph G,*

$$\text{egd}(G) = \max_{x \in \text{ext}\,\mathscr{E}(G)} \text{gd}(G, x). \tag{10.5}$$

Proof. If $\text{egd}(G) = r$, it follows by (10.4) that $\text{ext}\,\mathscr{E}(G) \subseteq \pi_E(\mathscr{E}_{n,r})$, which implies that $\max_{x \in \text{ext}\,\mathscr{E}(G)} \text{gd}(G, x) \leq r$. On the other hand, if $\max_{x \in \text{ext}\,\mathscr{E}(G)} \text{gd}(G, x) = r$, every element $x \in \text{ext}\,\mathscr{E}(G)$ has a psd completion of rank at most r and thus $\text{ext}\,\mathscr{E}(G) \subseteq \pi_E(\mathscr{E}_{n,r})$. □

10.1.1 Properties of the extreme Gram dimension

In this section we investigate the behavior of the graph parameter $\text{egd}(\cdot)$ under some simple graph operations: taking minors and clique sums.

10.1.3 Lemma. *The parameter* $\text{egd}(\cdot)$ *is minor monotone, i.e., for any edge e of G,*

$$\text{egd}(G \backslash e) \leq \text{egd}(G) \ \textit{ and } \ \text{egd}(G/e) \leq \text{egd}(G).$$

Proof. Consider a graph $G = ([n], E)$ with $\text{egd}(G) = r$ and let $e \in E$. We first show that $\text{egd}(G \setminus e) \leq \text{egd}(G) = r$. By Definition 10.1.1 it suffices to prove that $\mathscr{E}(G \setminus e) \subseteq \pi_{E \setminus e}(\text{conv}(\mathscr{E}_{n,r}))$. Let $x \in \mathscr{E}(G \setminus e)$ and choose a scalar $x_e \in [-1, 1]$ such that $(x, x_e) \in \mathscr{E}(G)$. Since $\text{egd}(G) = r$ it follows that $(x, x_e) \in \pi_E(\text{conv}(\mathscr{E}_{n,r}))$ and thus $x \in \pi_{E \setminus e}(\text{conv}(\mathscr{E}_{n,r}))$.

We now show that $\text{egd}(G/e) \leq \text{egd}(G) = r$. Say, $e = (n-1, n)$ and set $G/e = ([n-1], E^{'})$. By Definition 10.1.1 it suffices to prove that $\mathscr{E}(G/e) \subseteq \pi_{E'}(\text{conv}(\mathscr{E}_{n-1,r}))$. For this, let $x \in \mathscr{E}(G/e)$. Then $x = \pi_{E'}(X)$ for some matrix $X \in \mathscr{E}_{n-1}$. Let $X[\cdot, n-1]$ denote the last column of X and define the new matrix

$$Y = \begin{pmatrix} X & X[\cdot, n-1] \\ X[\cdot, n-1]^{\mathsf{T}} & 1 \end{pmatrix} \in \mathscr{S}^n.$$

By construction, we have that $Y \in \mathscr{E}_n$ and $Y_{n-1,n} = 1$. Moreover, define $y = \pi_E(Y)$ and notice that $y \in \mathscr{E}(G)$. Since $\mathrm{egd}(G) = r$, it follows from Definition 10.1.1 that there exist matrices $Y_1, \dots, Y_m \in \mathscr{E}_{n,r}$ and scalars $\lambda_i > 0$ with $\sum_{i=1}^m \lambda_i = 1$ satisfying

$$y = \pi_E(\sum_{i=1}^m \lambda_i Y_i). \tag{10.6}$$

Since $(n-1, n) \in E$, we have that $y_{n-1,n} = Y_{n-1,n} = 1$ and then (10.6) gives

$$1 = \sum_{i=1}^m \lambda_i (Y_i)_{n-1,n}. \tag{10.7}$$

Since the matrices Y_i $(i \in [m])$ are psd with diagonal entries equal to one, all their entries are bounded in absolute value by 1. Moreover we have that $\lambda_i \in (0,1)$ for every $i \in [m]$. Then, (10.7) implies that $(Y_i)_{n-1,n} = 1$ for all $i \in [m]$. Combining this with the fact that $(Y_i)_{n,n} = 1$ we get that $Y_i[\cdot, n-1] = Y_i[\cdot, n]$ for all $i \in [m]$.

For $i \in [m]$, let X_i be the matrix obtained from Y_i by removing its n-th row and its n-th column. Since X_i is a submatrix of Y_i we have that $\mathrm{rank} X_i \leq \mathrm{rank}\, Y_i \leq r$. Moreover, since $Y_i[\cdot, n-1] = Y_i[\cdot, n]$ for all $i \in [m]$ it follows that $x = \pi_{E'}(\sum_{i=1}^m \lambda_i X_i)$. Lastly, since $\sum_{i=1}^m \lambda_i X_i \in \mathrm{conv}(\mathscr{E}_{n-1,r})$ it follows that $x \in \pi_{E'}(\mathrm{conv}(\mathscr{E}_{n-1,r}))$. This concludes the proof that $\mathrm{egd}(G/e) \leq r$. $\square$

Recall that, if G is the clique sum of G_1 and G_2, its Gram dimension satisfies: $\mathrm{gd}(G) = \max\{\mathrm{gd}(G_1), \mathrm{gd}(G_2)\}$. For the extreme Gram dimension, the analogous result holds only for clique k-sums with $k \leq 1$.

10.1.4 Lemma. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. If $|V_1 \cap V_2| \leq 1$ then the clique sum G of G_1, G_2 satisfies* $\mathrm{egd}(G) = \max\{\mathrm{egd}(G_1), \mathrm{egd}(G_2)\}$.

Proof. Let $x \in \mathscr{E}(G)$ and set $r = \max\{\mathrm{egd}(G_1), \mathrm{egd}(G_2)\}$. We will show that $x \in \pi_E(\mathrm{conv}(\mathscr{E}_{n,r}))$. For $i = 1, 2$, the vector $x_i = \pi_{E_i}(x)$ belongs to $\pi_{E_i}(\mathrm{conv}(\mathscr{E}_{|V_i|,r}))$. Hence, $x_i = \pi_{E_i}(\sum_{j=1}^{m_i} \lambda_{i,j} X^{i,j})$ for some $X^{i,j} \in \mathscr{E}_{|V_i|,r}$ and $\lambda_{i,j} \geq 0$ with $\sum_j \lambda_{i,j} = 1$. As $|V_1 \cap V_2| \leq 1$, any two matrices $X^{1,j}$ and $X^{2,k}$ share at most one diagonal entry, equal to 1 in both matrices. By Lemma 2.3.11, $X^{1,j}$ and $X^{2,k}$ have a common completion $Y^{j,k} \in \mathscr{E}_{n,r}$. This implies that $x = \pi_E(\sum_{j=1}^{m_1} \sum_{k=1}^{m_2} \lambda_{1,j} \lambda_{2,k} Y^{jk})$, which shows $x \in \pi_E(\mathrm{conv}(\mathscr{E}_{n,r}))$. $\square$

Throughout this section we denote by $\mathscr{G}_r$ the class of graphs having extreme Gram dimension at most r. By Lemma 10.1.3 and Lemma 10.1.4 the class $\mathscr{G}_r$ is closed under taking disjoint unions and clique 1-sums of graphs. Nevertheless, it is *not* closed under clique k-sums when $k \geq 2$. E.g. the graph F_3 from Figure 10.1 is a clique 2-sum of triangles, however $\mathrm{egd}(F_3) = 3$ (Theorem 10.1.15) while triangles have extreme Gram dimension 2 (Lemma 10.1.5).

10.1.2 Showing upper and lower bounds

From Lemma 10.1.2 we know that for any graph G,

$$\mathrm{egd}(G) = \max_{x \in \mathrm{ext}\, \mathscr{E}(G)} \mathrm{gd}(G, x).$$

According to this characterization for the extreme Gram dimension, in order to show that $\mathrm{egd}(G) \leq r$, it suffices to show that every partial matrix $x \in \mathrm{ext}\, \mathscr{E}(G)$ has a psd completion of rank at most r. Using some well-known results concerning the ranks of extreme points of $\mathcal{E}_n$ we obtain the following:

10.1.5 Lemma. *The extreme Gram dimension of the complete graph K_n is*

$$\mathrm{egd}(K_n) = \max\left\{ r \in \mathbb{Z}_+ : \binom{r+1}{2} \leq n \right\} = \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor. \tag{10.8}$$

Hence, for any graph G on n nodes we have that:

$$\mathrm{egd}(G) \leq \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor. \tag{10.9}$$

Proof. Notice that $\mathcal{E}(K_n)$ is the bijective image of $\mathcal{E}_n$ in $\mathbb{R}^{\binom{n}{2}}$, obtained by considering only the upper triangular part of matrices in $\mathcal{E}_n$. Then, for any $X \in \mathrm{ext}\,\mathcal{E}_n$ with $\mathrm{rank} X = r$ we know that $\binom{r+1}{2} \leq n$ (recall (3.11)). Moreover, from Theorem 3.2.7 we know that for any natural number r satisfying $\binom{r+1}{2} \leq n$ there exists an extreme point of $\mathcal{E}_n$ with rank equal to r. □

The following lemma is a direct consequence of (10.8).

10.1.6 Lemma. *Consider a graph G with $|V(G)| = \binom{r'+1}{2}$. Then* $\mathrm{egd}(G) \leq r'$.

On the other hand, to obtain a lower bound $\mathrm{egd}(G) \geq r$, it follows from (10.5) that we need a vector $x \in \mathrm{ext}\,\mathcal{E}(G)$, all of whose positive semidefinite completions have rank at least $r+1$. This approach presents two main challenges: First of all, we do not know of any way to generate extreme elements of $\mathcal{E}(G)$. Moreover, even if we are given a point in $\mathrm{ext}\,\mathcal{E}(G)$, it is not clear how to verify that all its positive semidefinite completions have rank at least $r+1$.

Our approach for showing lower bounds is to construct elements in $\mathcal{E}(G)$ that admit a unique completion to a full positive semidefinite matrix. For such partial matrices the problem of showing that all its psd completions have rank at least $r+1$ is easy (there is only 1 completion to check). If we additionally assume that this unique completion is an extreme point of $\mathcal{E}_n$, then its projection is an extreme point of $\mathcal{E}(G)$. This approach is summarized in the following lemma.

10.1.7 Lemma. *Suppose that there exists $x \in \mathcal{E}(G)$ such that* $\mathrm{fib}(x) = \{X\}$ *where $X \in \mathrm{ext}\,\mathcal{E}_n$ and* $\mathrm{rank} X = r$. *Then* $\mathrm{egd}(G) \geq r$.

Proof. As $\mathrm{fib}(x) = \{X\}$ and $X \in \mathrm{ext}\,\mathcal{E}_n$, it follows that $\mathrm{fib}(x)$ is a face of $\mathcal{E}_n$ and then Lemma 4.3.9 implies that $x \in \mathrm{ext}\,\mathcal{E}(G)$. □

10.1.3 The strong largeur d'arborescence

In this section we introduce a new width parameter that will serve as an upper bound for the extreme Gram dimension.

10.1.8 Definition. *The* strong largeur d'arborescence *of a graph G, denoted by $\mathrm{la}_{\boxtimes}(G)$, is the smallest integer $k \geq 1$ for which G is a minor of $T \boxtimes K_k$ for some tree T.*

Notice the analogy with the largeur d'arborescence (cf. Definition 2.2.4) where we have substituted the Cartesian product with the strong graph product. It is clear from its definition that the parameter $\mathrm{la}_{\boxtimes}(\cdot)$ is minor monotone. Moreover,

10.1.9 Lemma. *For any graph G we have that*

$$\frac{\mathrm{tw}(G)+1}{2} \leq \mathrm{la}_{\boxtimes}(G) \leq \mathrm{la}_{\square}(G).$$

Proof. The rightmost inequality follows directly from the definitions. For the leftmost inequality assume that $\mathrm{la}_{\boxtimes}(G) = k$, i.e., G is minor of $T \boxtimes K_k$ for some tree T. Notice that the graph $T \boxtimes K_k$ can be obtained by clique k-sums of the graph $K_2 \boxtimes K_k$. Combining the fact that the treewidth of a graph is a minor-monotone graph parameter with Lemma 2.2.3 the claim follows. $\square$

Our main goal in this section is to show that the extreme Gram dimension is upper bounded by the strong largeur d'arborescence: $\mathrm{egd}(G) \leq \mathrm{la}_{\boxtimes}(G)$ for any graph G. As we will see in later sections, the class of graphs with $\mathrm{la}_{\boxtimes}(G) \leq 2$ plays a crucial role in characterizing graphs with extreme gram dimension at most 2. We start with a useful lemma.

10.1.10 Lemma. *Let $\{u_1, \dots, u_{2r}\}$ be a set of vectors, denote its rank by ρ. Let $\mathscr{U}$ denote the linear span of the matrices $U_{ij} = (u_i u_j^\mathsf{T} + u_j u_i^\mathsf{T})/2$ for all $i, j \in \{1, \dots, r\}$ and all $i, j \in \{r+1, \dots, 2r\}$. If $\rho \geq r+1$ then $\dim \mathscr{U} < \binom{\rho+1}{2}$.*

Proof. Let $I \subseteq \{1, \dots, r\}$ for which $\{u_i : i \in I\}$ is a maximum linearly independent subset of $\{u_1, \dots, u_r\}$ and let $J \subseteq \{r+1, \dots, 2r\}$ such that the set $\{u_i : i \in I \cup J\}$ is maximum linearly independent; thus $|I| + |J| = \rho$. Set $K = \{1, \dots, r\} \setminus I$, $L = \{r+1, \dots, 2r\} \setminus J$, and $J' = J \setminus \{k\}$, where k is some given (fixed) element of J. For any $l \in L$, there exists scalars $a_{l,i} \in \mathbb{R}$ such that

$$u_l = \sum_{i \in I \cup J'} a_{l,i} u_i + a_{l,k} u_k. \tag{10.10}$$

Set

$$A_l = \sum_{i \in I \cup J'} a_{l,i} U_{ik} \quad \text{for } \; l \in L.$$

Then, define the set $\mathscr{W}$ consisting of the matrices U_{ii} for $i \in I \cup J$, U_{ij} for all $i \neq j$ in $I \cup J'$, U_{kj} for all $j \in J'$, and A_l for all $l \in L$. Then, $|\mathscr{W}| = \rho + \binom{\rho-1}{2} + r - 1 = \binom{\rho}{2} + r = \binom{\rho+1}{2} + r - \rho \leq \binom{\rho+1}{2} - 1$. In order to conclude the proof it suffices to show that $\mathscr{W}$ spans the space $\mathscr{U}$.

Clearly, $\mathscr{W}$ spans all matrices U_{ij} with $i, j \in \{1, \dots, r\}$. Moreover, by its definition $\mathscr{W}$ contains all matrices U_{ij} for $i, j \in J$. Consequently, it remains to show that $U_{kl} \in \mathscr{W}$ for all $l \in L$, $U_{lj} \in \mathscr{W}$ for all $l \in L$ and $j \in J'$ and that $U_{ll'} \in \mathscr{W}$ for all $l, l' \in L$. Fix $l \in L$. Using (10.10) we obtain that $U_{lk} = A_l + a_{l,k} U_{kk}$ lies in the span of $\mathscr{W}$. Moreover, for $j \in J'$, $U_{lj} = \sum_{i \in I \cup J'} a_{l,i} U_{ij} + a_{l,k} U_{kj}$ also lies in the span of $\mathscr{W}$. Finally, for $l' \in L$, $U_{ll'} = \sum_{i,j \in I \cup J'} a_{l,i} a_{l',j} U_{ij} + a_{l',k} A_l + a_{l,k} A_{l'} + a_{l,k} a_{l',k} U_{kk}$ is also spanned by $\mathscr{W}$. This concludes the proof. $\square$

10.1.11 Lemma. *Let $v_1, \dots, v_n$ be a family of linearly independent vectors in $\mathbb{R}^n$. Then the matrices $(v_i v_j^\mathsf{T} + v_j v_i^\mathsf{T})$ for $1 \leq i \leq j \leq n$ span $\mathscr{S}^n$.*

Proof. Consider a matrix $Z \in \mathscr{S}^n$ such that $\langle Z, (v_i v_j^\mathsf{T} + v_j v_i^\mathsf{T})/2 \rangle = 0$ for all $i, j \in [n]$. We will show that Z is the zero matrix. For a vector $x \in \mathbb{R}^n$ we have that $x = \sum_{i=1}^n \lambda_i v_i$ for some scalars λ_i ($i \in [n]$) and thus $x^\mathsf{T} Z x = 0$. This implies that Z is the zero matrix. $\square$

Using the Lemma 10.1.10 we can now obtain the main result in this section.

10.1.12 Theorem. *For any tree T we have that* $\mathrm{egd}(T \boxtimes K_r) \le r$.

Proof. Let $G = T \boxtimes K_r$, where T is a tree on $[t]$ and let $G = (V, E)$ with $|V| = n$. So the node set of G is $V = \cup_{i=1}^{t} V_i$, where the V_i's are pairwise disjoint sets, each of cardinality r. By definition of the strong product, for any edge $\{i, j\}$ of T, the set $V_i \cup V_j$ induces a clique in G, denoted as C_{ij}. Then, G is the union of the cliques C_{ij} over all edges $\{i, j\}$ of T. We show that $\mathrm{egd}(G) \le r$. For this, pick an extreme element $x \in \mathrm{ext}\,\mathscr{E}(G)$. Then $x = \pi_E(X)$ for some $X \in \mathscr{E}_n$. As C_{ij} is a clique in G, the principal submatrix $X^{ij} := X[C_{ij}]$ is fully determined from x. In order to show that x has a psd completion of rank at most r, it suffices to show that $\mathrm{rank} X^{ij} \le r$ for all edges $\{i, j\}$ of T (then apply Lemma 2.3.11).

Pick an edge $\{i, j\}$ of T and set $\rho = \mathrm{rank} X^{ij}$. Assume that $\rho \ge r + 1$; we show below that there exists a nonzero perturbation Z of X^{ij} such that

$$\begin{aligned} &Z_{hk} = 0 \ \forall (h,k) \in (V_i \times V_i) \cup (V_j \times V_j), \\ &Z_{hk} \neq 0 \text{ for some } (h,k) \in V_i \times V_j. \end{aligned} \tag{10.11}$$

This permits to reach a contradiction: As Z is a perturbation of X^{ij}, there exists $\epsilon > 0$ for which $X^{ij} + \epsilon Z$, $X^{ij} - \epsilon Z \succeq 0$. By construction, C_{ij} is the only maximal clique of G containing the edges $\{h, k\}$ of G with $h \in V_i$ and $k \in V_j$. Hence, one can find a psd completion X' (resp., X'') of the matrix $X^{ij} + \epsilon Z$ (resp., $X^{ij} - \epsilon Z$) and the matrices $X^{i'j'}$ for all edges $\{i', j'\} \neq \{i, j\}$ of T. Now, $x = \frac{1}{2}(\pi_E(X') + \pi_E(X''))$, where $\pi_E(X'), \pi_E(X'')$ are distinct elements of $\mathscr{E}(G)$, contradicting the fact that x is an extreme point of $\mathscr{E}(G)$.

We now construct the desired perturbation Z of X^{ij} satisfying (10.11). For this let u_h ($h \in V_i \cup V_j$) be a Gram representation of X^{ij} in $\mathbb{R}^\rho$ and let $\mathscr{U} \subseteq \mathscr{S}_\rho$ denote the linear span of the matrices $U_{hk} = (u_h u_k^{\mathsf{T}} + u_k u_h^{\mathsf{T}})/2$ for all $h, k \in V_i$ and all $h, k \in V_j$. Applying Lemma 10.1.10, as $\rho \ge r + 1$, we deduce that $\dim \mathscr{U} < \binom{\rho+1}{2}$. Hence there exists a nonzero matrix $R \in \mathscr{S}_\rho$ lying in $\mathscr{U}^\perp$. Define the matrix $Z \in \mathscr{S}_{2r}$ by $Z_{hk} = \langle R, U_{hk} \rangle$ for all $h, k \in V_i \cup V_j$. By construction, Z is a perturbation of X^{ij} (recall Proposition 3.2.6 and it satisfies $Z_{hk} = 0$ whenever the pair (h, k) is contained in V_i or in V_j. Moreover, as $R \neq 0$ Lemma 10.1.11 implies that Z is a nonzero matrix and thus $Z_{hk} \neq 0$ for some $h \in V_i$ and $k \in V_j$. Thus (10.11) holds and the proof is completed. □

10.1.13 Corollary. *For any graph G,* $\mathrm{egd}(G) \le \mathrm{la}_{\boxtimes}(G)$.

Proof. If $\mathrm{la}_{\boxtimes}(G) = k$, then G is a minor of $T \boxtimes K_k$ for some tree T and thus $\mathrm{egd}(G) \le \mathrm{egd}(T \boxtimes K_k) \le k$, by Lemma 10.1.3 and Theorem 10.1.12. □

10.1.4 Graph families with unbounded extreme Gram dimension

In this section we construct three classes of graphs F_r, G_r, H_r, whose extreme Gram dimension is equal to r. Therefore, they are forbidden minors for the class $\mathscr{G}_{r-1}$ of graphs with extreme Gram dimension at most $r - 1$. As we will see in the next section, this gives all the forbidden minors for the characterization of the class $\mathscr{G}_2$.

The graphs G_r were already considered by Colin de Verdière [43] in relation to the graph parameter $\nu(\cdot)$, to which we will come back in Section 10.3. Each of the graphs $G = F_r, G_r, H_r$ has $\binom{r+1}{2}$ nodes and thus extreme Gram dimension at most r (recall Lemma 10.1.6). Similarly, $\text{egd}(G/e) \le r-1$ after contracting an edge. To show equality $\text{egd}(G) = r$, we rely on Lemma 10.1.7.

To use Lemma 10.1.7 we need tools permitting to show existence of a *unique* completion for a vector $x \in \mathscr{E}(G)$. We introduce below such a tool: 'forcing a non-edge with a minimally singular clique', based on the following property of psd matrices:

$$\begin{pmatrix} A & b \\ b^{\mathsf{T}} & \alpha \end{pmatrix} \succeq 0 \Longrightarrow b^{\mathsf{T}} u = 0 \ \forall u \in \operatorname{Ker} A. \tag{10.12}$$

10.1.14 Lemma. *Let $x \in \mathscr{E}(G)$, let $C \subseteq V$ be a clique of G and $\{i,j\} \notin E(G)$ with $i \notin C$, $j \in C$. Set $x[C] = (x_{ij})_{i,j\in C} \in \mathscr{E}_{|C|}$ (setting $x_{ii} = 1$ for all i). Assume that i is adjacent to all nodes of $C \setminus \{j\}$ and that $x[C]$ is minimally singular (i.e., $x[C]$ is singular but any principal submatrix of $x[C]$ is nonsingular). Then the (i,j)-th entry X_{ij} is uniquely defined in any completion $X \in \text{fib}(x)$ of x.*

Proof. Let $X \in \text{fib}(x)$. The principal submatrix $X[C \cup \{i\}]$ has the block form shown in (10.12) where all entries are specified (from x) except the entry $b_j = X_{ij}$. As $x[C]$ is singular there exists a nonzero vector u in the kernel of $x[C]$. Moreover, $u_j \neq 0 \quad \forall j \in C$, since $x[C \setminus \{j\}]$ is nonsingular. Hence the condition $b^{\mathsf{T}} u = 0$ permits to derive the value of X_{ij} from x. □

When applying Lemma 10.1.14 we will say that "the clique C forces the pair $\{i,j\}$". The lemma will be used in an iterative manner: Once a non-edge $\{i,j\}$ has been forced, we know the value X_{ij} in any psd completion X and thus we can replace G by $G + \{i,j\}$ and search for a new forced pair in the extended graph $G + \{i,j\}$.

The class F_r

For $r \ge 2$ the graph F_r has $r + \binom{r}{2} = \binom{r+1}{2}$ nodes, denoted as v_i (for $i \in [r]$) and v_{ij} (for $1 \le i < j \le r$); it consists of a clique K_r on the nodes $\{v_1, \dots, v_r\}$ together with the cliques C_{ij} on $\{v_i, v_j, v_{ij}\}$ for all $1 \le i < j \le r$. The graphs F_3 and F_4 are illustrated in Figure 10.1.

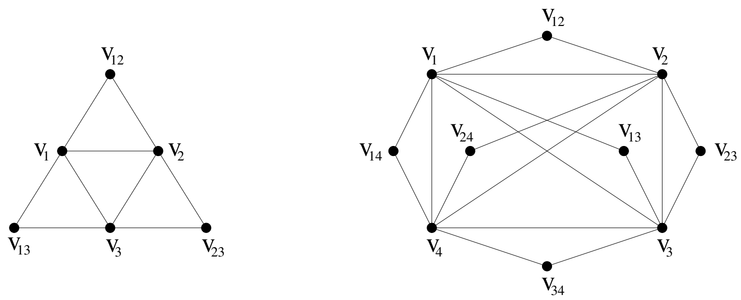

Figure 10.1: The graphs F_3 and F_4.

For $r = 2$, $F_2 = K_3$ has extreme Gram dimension 2. More generally:

10.1.15 Theorem. *For $r \ge 2$, $\text{egd}(F_r) = r$. Moreover, F_r is a minimal forbidden minor for the class $\mathscr{G}_{r-1}$.*

Proof. First we show that $\mathrm{egd}(F_r) \geq r$. For this we label the nodes $v_1, \ldots, v_r$ by the standard unit vectors $e_1, \ldots, e_r \in \mathbb{R}^r$ and v_{ij} by the vector $(e_i + e_j)/\sqrt{2}$. Consider the Gram matrix X of these $n = \binom{r+1}{2}$ vectors and its projection $x = \pi_{E(F_r)}(X) \in \mathscr{E}(F_r)$. Using (3.10) it follows directly that X is an extreme point of $\mathscr{E}_n$. We now show that X is the only psd completion of x which, in view of Lemma 10.1.7, implies that $\mathrm{egd}(F_r) \geq r$. For this we use Lemma 10.1.14. Observe that, for each $1 \leq i < j \leq r$, the matrix $x[C_{ij}]$ is minimally singular. First, for any $k \in [r] \setminus \{i, j\}$, the clique C_{ij} forces the non-edge $\{v_k, v_{ij}\}$ and then, for any other $1 \leq i' < j' \leq r$, the clique C_{ij} forces the non-edge $\{v_{ij}, v_{i'j'}\}$. Hence, in any psd completion of x, all the entries indexed by non-edges are uniquely determined, i.e., $\mathrm{fib}(x) = \{X\}$.

Next, we show minimality. Let e be an edge of F_r, we show that $\mathrm{egd}(H) \leq r - 1$ where $H = F_r \backslash e$. If e is an edge of the form $\{v_i, v_{ij}\}$, then H is the clique 1-sum of an edge and a graph on $\binom{r+1}{2} - 1$ nodes and thus $\mathrm{egd}(H) \leq r - 1$ follows using Lemmas 10.1.4 and 10.1.5. Suppose now that e is contained in the central clique K_r, say $e = \{v_1, v_2\}$. We show that H is contained in a graph of the form $T \boxtimes K_{r-1}$ for some tree T. We choose T to be the star $K_{1,r-1}$ and we give a suitable partition of the nodes of F_r into sets $V_0 \cup V_1 \cup \ldots \cup V_{r-1}$, where each V_i has cardinality at most $r - 1$, V_0 is assigned to the center node of the star $K_{1,r-1}$ and $V_1, \ldots, V_{r-1}$ are assigned to the $r - 1$ leaves of $K_{1,r-1}$. Namely, set $V_0 = \{v_{12}, v_3, \ldots, v_r\}$, $V_1 = \{v_1, v_{13}, \ldots, v_{1r}\}$, $V_2 = \{v_2, v_{23}, \ldots, v_{2r}\}$ and, for $k \in \{3, \ldots, r-1\}$, $V_k = \{v_{kj} : k+1 \leq j \leq r\}$. Then, in the graph H, each edge is contained in one of the sets $V_0 \cup V_k$ for $1 \leq k \leq r - 1$. This shows that H is a subgraph of $K_{1,r-1} \boxtimes K_{r-1}$ and thus $\mathrm{egd}(H) \leq r - 1$ (by Theorem 10.1.12). □

As an application of Theorem 10.1.15 we get:

10.1.16 Corollary. *If the tree T has a node of degree at least $(r-1)/2$ then* $\mathrm{egd}(T \boxtimes K_r) = r$.

Proof. Directly from Theorem 10.1.15, as $T \boxtimes K_r$ contains a subgraph F_r. □

The class G_r

Consider an equilateral triangle and subdivide each side into $r - 1$ equal segments. Through these points draw line segments parallel to the sides of the triangle. This construction creates a triangulation of the big triangle into $(r-1)^2$ congruent equilateral triangles. The graph G_r corresponds to the edge graph of this triangulation. The graph G_5 is illustrated in Figure 10.2.

The graph G_r has $\binom{r+1}{2}$ vertices, which we denote $v_{i,l}$ for $l \in [r]$ and $i \in [r-l+1]$ (with $v_{1,l}, \ldots, v_{r-l+1,l}$ at level l, see Figure 10.2). Note that $G_2 = K_3 = F_2$, $G_3 = F_3$, but $G_r \neq F_r$ for $r \geq 4$. Using the following lemma we can construct some points of $\mathscr{E}(G_r)$ with a unique completion.

10.1.17 Lemma. *Consider a labeling of the nodes of G_r by vectors $w_{i,l}$ satisfying the following property (P_r): For each triangle $C_{i,l} = \{v_{i,l}, v_{i+1,l}, v_{i,l+1}\}$ of G_r, the set $\{w_{i,l}, w_{i+1,l}, w_{i,l+1}\}$ is minimally linearly dependent. (These triangles are shaded in Figure 10.2). Let X be the Gram matrix of the vectors $w_{i,l}$ and let $x = \pi_{E(G_r)}(X)$ be its projection. Then X is the unique completion of x.*

Proof. For $r = 2$, $G_2 = K_3$ and there is nothing to prove. Let $r \geq 3$ and assume that the claim holds for $r - 1$. Consider a labeling $w_{i,l}$ of G_r satisfying (P_r) and the

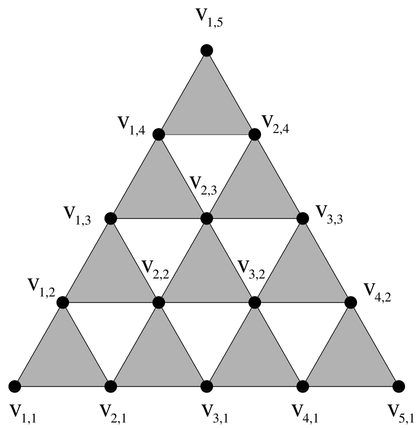

Figure 10.2: The graph G_5.

corresponding vector $x \in \mathscr{E}(G_r)$. We show, using Lemma 10.1.14, that the entries Y_{uv} of a psd completion Y of x are uniquely determined for all $\{u,v\} \notin E(G_r)$. For this, denote by H, R, L the sets of nodes lying on the 'horizontal' side, the 'right' side and the 'left' side of G_r, respectively (refer to the drawing of G_r of Figure 10.2). Observe that each of $G_r \backslash H$, $G_r \backslash R$, $G_r \backslash L$ is a copy of G_{r-1}. As the induced vector labelings on each of these graphs satisfies the property (P_{r-1}), we can conclude using the induction assumption that the entry Y_{uv} is uniquely determined whenever the pair $\{u,v\}$ is contained in the vertex set of one of $G_r \backslash H$, $G_r \backslash R$, or $G_r \backslash L$. The only non-edges $\{u,v\}$ that are not yet covered arise when u is a corner of G_r and v lies on the opposite side, say $u = v_{1,1}$ and $v = v_{r-l+1,l} \in R$. If $l \neq 1, r$ then the clique $C_{1,1} = \{v_{1,1}, v_{2,1}, v_{1,2}\}$ forces the pair $\{u,v\}$ (since $\{v, v_{1,2}\} \subseteq E(G_r \backslash H)$ and $\{v, v_{2,1}\} \subseteq E(G_r \backslash L)$). If $l = r$ then the clique $C_{1,r-1} = \{v_{1,r-1}, v_{2,r-1}, v_{1,r}\}$ forces the pair $\{u,v\}$ (since $\{u, v_{1,r-1}\} \subseteq E(G_r \backslash R)$ and the value at the pair $\{u, v_{2,r-1}\}$ has just been specified). Analogously for the case $l = 1$. This concludes the proof. □

10.1.18 Theorem. *We have that* $\mathrm{egd}(G_r) = r$ *for all* $r \geq 2$. *Moreover,* G_r *is a minimal forbidden minor for the class* $\mathscr{G}_{r-1}$.

Proof. We first show that $\mathrm{egd}(G_r) \geq r$. For this, choose a vector labeling of the nodes of G_r satisfying the conditions of Lemma 10.1.17: Label the nodes $v_{1,1}, \ldots, v_{r,1}$ at level $l = 1$ by the standard unit vectors $w_{1,1} = e_1, \ldots, w_{r,1} = e_r$ in $\mathbb{R}^r$ and define inductively $w_{i,l+1} = \frac{w_{i,l} + w_{i+1,l}}{\|w_{i,l} + w_{i+1,l}\|}$ for $l = 1, \ldots, r-1$. By Lemma 10.1.17 their Gram matrix X is the unique completion of its projection $x = \pi_{E(G_r)}(X) \in \mathscr{E}(G_r)$. Moreover, X is extreme in $\mathscr{E}_n$ since $\mathscr{U}_V$ is full-dimensional in $\mathscr{S}^r$. This shows $\mathrm{egd}(G_r) \geq r$, by Lemma 10.1.7.

We now show that $\mathrm{egd}(G_r \backslash e) \leq r-1$. For this use the inequalities: $\mathrm{egd}(G_r \backslash e) \leq \mathrm{la}_{\boxtimes}(G_r \backslash e) \leq \mathrm{la}_{\square}(G_r \backslash e) \leq r-1$, where the leftmost inequality follows from Corollary 10.1.13 and the rightmost one is shown in [70]. □

We conclude with two immediate corollaries.

10.1.19 Corollary. *The graph parameter* $\mathrm{egd}(G)$ *is unbounded for the class of planar graphs.*

10.1.20 Corollary. *Let T be a tree which contains a path with $2r-2$ nodes. Then,* $\mathrm{egd}(T \boxtimes K_r) = r$.

Proof. It is shown in [43] that G_r is a minor of the Cartesian product of two paths P_r and P_{2r-2} (with, respectively, r and $2r-2$ nodes). Hence, $G_r \preceq P_{2r-2} \square P_r \preceq T \boxtimes K_r$ and thus $r = \mathrm{egd}(G_r) \le \mathrm{egd}(T \boxtimes K_r)$. □

The class H_r

In this section we consider a third class of graphs H_r for every $r \ge 3$. In order to explain the general definition we first describe the base case $r = 3$.

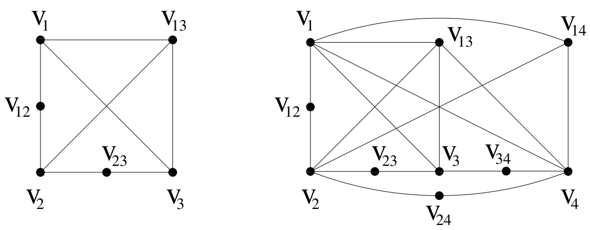

Figure 10.3: The graphs H_3 and H_4.

The graph H_3 is shown in Figure 10.3. It is obtained by taking a complete graph K_4, with vertices v_1, v_2, v_3 and v_{13}, and subdividing two adjacent edges: here we insert node v_{12} between v_1 and v_2 and node v_{23} between nodes v_2 and v_3.

10.1.21 Lemma. $\mathrm{egd}(H_3) = 3$ *and H_3 is a minimal forbidden minor for $\mathcal{G}_2$.*

Proof. As H_3 has 6 nodes, $\mathrm{egd}(H_3) \le 3$. To show equality, we use the following vector labeling for the nodes of H_3: Label the nodes v_1, v_2, v_3 by the standard unit vectors $e_1, e_2, e_3 \in \mathbb{R}^3$ and v_{ij} by $(e_i + e_j)/\sqrt{2}$ for $1 \le i < j \le 3$. Let $X \in \mathcal{E}_6$ be their Gram matrix and set $x = \pi_{E(H_3)}(X) \in \mathcal{E}(H_3)$. Then X has rank 3 and X is an extreme point of $\mathcal{E}_6$. We now show that X is the unique completion of x in $\mathcal{E}_6$. For this let $Y \in \mathrm{fib}(x)$. Consider its principal submatrices Z, Z' indexed by $\{v_1, v_2, v_3, v_{13}\}$ and $\{v_1, v_2, v_{12}\}$, of the form:

$$Z = \begin{pmatrix} 1 & a & 0 & \sqrt{2}/2 \\ a & 1 & b & 0 \\ 0 & b & 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 & 1 \end{pmatrix} \qquad Z' = \begin{pmatrix} 1 & a & \sqrt{2}/2 \\ a & 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \end{pmatrix},$$

where $a, b \in \mathbb{R}$. Then, $\det(Z) = -(a+b)^2/2$ implies $a + b = 0$, and $\det(Z') = a(1-a)$ implies $a \ge 0$. Similarly, $b \ge 0$ using the principal submatrix of Y indexed by $\{v_2, v_3, v_{23}\}$. This shows $a = b = 0$ and thus the entries of Y at the positions $\{v_1, v_2\}$ and $\{v_2, v_3\}$ are uniquely specified. Remains to show that the entries are uniquely specified at the non-edges containing v_{12} or v_{23}. For this we use Lemma 10.1.14: First the clique $\{v_2, v_3, v_{23}\}$ forces the pairs $\{v_1, v_{23}\}$ and $\{v_{13}, v_{23}\}$ and then the clique $\{v_1, v_2, v_{12}\}$ forces the pairs $\{v_{23}, v_{12}\}$, $\{v_{13}, v_{12}\}$, and $\{v_3, v_{12}\}$. Thus we have shown $Y = X$, which concludes the proof that $\mathrm{egd}(H_3) = 3$.

We now verify that it is a minimal forbidden minor. Contracting any edge results in a graph on 5 nodes and we are done (10.9). Lastly, we verify that $\mathrm{egd}(H_3 \backslash e) \le 2$

for each edge $e \in E(H_3)$. If deleting the edge e creates a cut node, then the result follows using Lemma 10.1.4. Otherwise, $H_3 \backslash e$ is contained in $T \boxtimes K_2$, where T is a path (for $e = \{v_2, v_{13}\}$) or a claw $K_{1,3}$ (for $e = \{v_1, v_{13}\}$ or $\{v_3, v_{13}\}$), and the result follows from Theorem 10.1.12. $\square$

We now describe the graph H_r, or rather a class $\mathscr{H}_r$ of such graphs. Any graph $H_r \in \mathscr{H}_r$ is constructed in the following way. Its node set is $V = V_0 \cup V_3 \cup \ldots \cup V_r$, where $V_0 = \{v_{ij} : 3 \le i < j \le r\}$ and, for $i \in \{3, \ldots, r\}$, $V_i = \{v_1, v_2, v_{12}, v_i, v_{1i}, v_{2i}\}$. So H_r has $n = \binom{r+1}{2}$ nodes. Its edge set is defined as follows: On each set V_i we put a copy of H_3 (with index i playing the role of index 3 in the description of H_3 above) and, for each $3 \le i < j \le r$, we have the edges $\{v_i, v_{ij}\}$ and $\{v_j, v_{ij}\}$ as well as exactly one edge, call it e_{ij}, from the set

$$F_{ij} = \{\{v_i, v_j\}, \{v_i, v_{1j}\}, \{v_j, v_{1i}\}, \{v_{1i}, v_{1j}\}\}. \tag{10.13}$$

Figure 10.3 shows the graph H_4 for the choice $e_{34} = \{v_4, v_{13}\}$.

10.1.22 Theorem. *For any graph $H_r \in \mathscr{H}_r$ ($r \ge 3$), $\operatorname{egd}(H_r) = r$.*

Proof. We label the nodes $v_1, \ldots, v_r$ by $e_1, \ldots, e_r \in \mathbb{R}^r$ and v_{ij} by $(e_i + e_j)/\sqrt{2}$. Let $X \in \mathscr{E}_n$ be their Gram matrix and $x = \pi_{E(H_r)}(X) \in \mathscr{E}(H_r)$. Then X is an extreme point of $\mathscr{E}_n$, we show that $\operatorname{fib}(x) = \{X\}$. For this let $Y \in \operatorname{fib}(x)$. We already know that $Y[V_i] = X[V_i]$ for each $i \in \{3, \ldots, r\}$. Indeed, as the subgraph of H_r induced by V_i is H_3, this follows from the way we have chosen the labeling and from the proof of Lemma 10.1.21. Hence we may now assume that we have a complete graph on each V_i and it remains to show that the entries of Y are uniquely specified at the non-edges that are not contained in some set V_i ($3 \le i \le r$). For this note that the vectors labeling the set $C_{ij} = \{v_i, v_j, v_{ij}\}$ are minimally linearly dependent. Using Lemma 10.1.14, one can verify that all remaining non-edges are forced using these sets C_{ij} and thus $Y = X$. This shows that $\operatorname{egd}(H_r) \ge r$. $\square$

In contrast to the graphs F_r and G_r, we do not know whether $H_r \in \mathscr{H}_r$ is a *minimal* forbidden minor for $\mathscr{G}_{r-1}$ for $r \ge 4$.

10.1.5 Two special graphs: $K_{3,3}$ and K_5

In this section we consider the graphs $K_{3,3}$ and K_5 which will play a special role in the characterization of the class $\mathscr{G}_2$. First we compute the extreme Gram dimension of $K_{3,3}$. Note that its Gram dimension is $\operatorname{gd}(K_{3,3}) = 4$ as $K_{3,3}$ contains a K_4-minor but it contains no K_5 and $K_{2,2,2}$-minor; cf. Theorem 5.3.2.

Our main goal in this section is to show that the extreme Gram dimension of the graph $K_{3,3}$ is equal to 2, i.e., for any $x \in \operatorname{ext} \mathscr{E}(K_{3,3})$ there exists a psd completion of rank at most 2. We start by showing that any completion of an element of $\operatorname{ext} \mathscr{E}(K_{3,3})$ has rank at most 3.

10.1.23 Lemma. *For $x \in \operatorname{ext} \mathscr{E}(K_{3,3})$, any $X \in \operatorname{fib}(x)$ has rank at most 3.*

Proof. Let $x \in \operatorname{ext} \mathscr{E}(K_{3,3})$ and let $X \in \operatorname{fib}(x)$ with $\operatorname{rank} X \ge 4$. Let $u_1, \ldots, u_6$ be a Gram representation of X and choose a subset $\{u_i : i \in I\}$ of linearly independent vectors with $|I| = 4$. Let E_I denote the set of edges of $K_{3,3}$ induced by I and set

$$\mathscr{U}_I = \{U_{ii} : i \in I\} \cup \{U_{ij} : \{i, j\} \in E_I\}.$$

Then $\mathscr{U}_I$ consists of linearly independent elements; cf. Lemma 10.1.11. By Lemma 4.3.8, $\mathscr{U}_I$ is contained in $\{\mathscr{U}_{ii} : i \in [6]\}$ and thus it has dimension at most 6. On the other hand, as any four nodes induce at least three edges in $K_{3,3}$, we have that $|\mathscr{U}_I| \geq 4+3=7$ and thus the dimension of $\mathscr{U}_I$ is at least 7, a contradiction. $\square$

The proof of the main theorem relies on the following two lemmas.

10.1.24 Lemma. *Let $X, Z \in \mathscr{S}^n$ with $X \succeq 0$ and satisfying:*

$$Xz = 0 \Longrightarrow z^{\mathsf{T}} Z z \geq 0, \quad Xz = 0, z^{\mathsf{T}} Z z = 0 \Longrightarrow Zz = 0. \tag{10.14}$$

Then $X + tZ \succeq 0$ for some $t > 0$.

Proof. Up to an orthogonal transformation we may assume $X = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, where D is a diagonal matrix with positive diagonal entries. Correspondingly, write Z in block form: $Z = \begin{pmatrix} A & B \\ B^{\mathsf{T}} & C \end{pmatrix}$. The conditions (10.14) show that $C \succeq 0$ and that the kernel of C is contained in the kernel of B. This implies that $X + tZ \succeq 0$ for some $t > 0$. $\square$

10.1.25 Lemma. *Let $x \in \operatorname{ext} \mathscr{E}(K_{3,3})$, let $X \in \operatorname{ext} \operatorname{fib}(x)$ with* $\operatorname{rank} X = 3$ *and Gram representation $\{u_1, \ldots, u_6\} \subseteq \mathbb{R}^3$. Let $V_1 = \{1,2,3\}$ and $V_2 = \{4,5,6\}$ be the bipartition of the node set of $K_{3,3}$. There exist matrices $Y_1, Y_2 \in \mathscr{S}^3$ such that $Y_1 + Y_2 \succ 0$ and*

$$\langle Y_k, U_{ii} \rangle = 0 \ \forall i \in V_k \ \forall k \in \{1,2\} \ \textit{ and } \ \exists k \in \{1,2\} \ \exists i,j \in V_k \ \langle Y_k, U_{ij} \rangle \neq 0.$$

Proof. Define $\mathscr{U}_k = \langle U_{ii} : i \in V_k \rangle \subseteq \mathscr{W}_k = \langle U_{ij} : i,j \in V_k \rangle \subseteq \mathscr{S}^3$ for $k = 1,2$. With this notation we are looking for two matrices Y_1, Y_2 such that $Y_1 + Y_2 \succ 0, Y_1 \in \mathscr{U}_1^{\perp}, Y_2 \in \mathscr{U}_2^{\perp}$ and either $Y_1 \notin \mathscr{W}_1^{\perp}$ or $Y_2 \notin \mathscr{W}_2^{\perp}$.

Since $x \in \operatorname{ext} \mathscr{E}(K_{3,3})$ by Lemma 4.3.9 it follows that $\operatorname{fib}(x)$ is a face of $\mathscr{E}_n$ and by Lemma 2.1.5 we have that $X \in \operatorname{ext} \mathscr{E}_6$. Then (3.10) implies that $\dim \langle U_{ii} : i \in [6] \rangle = 6$ and thus $\dim \mathscr{U}_1 = \dim \mathscr{U}_2 = 3$. This implies that $\mathscr{U}_1 \cap \mathscr{U}_2 = \{0\}$ and thus $\mathscr{U}_1^{\perp} \cup \mathscr{U}_2^{\perp} = \mathscr{S}^3$. Moreover, as $\dim \mathscr{U}_1^{\perp} = \dim \mathscr{U}_2^{\perp} = 3$ it follows that $\mathscr{U}_1^{\perp} \cap \mathscr{U}_2^{\perp} = \{0\}$ and thus $\mathscr{S}^3 = \mathscr{U}_1^{\perp} \oplus \mathscr{U}_2^{\perp}$. Lastly, we have that $\mathscr{W}_k^{\perp} \subseteq \mathscr{U}_k^{\perp}$ $(k = 1,2)$ and thus $\mathscr{W}_1^{\perp} \cap \mathscr{W}_2^{\perp} \subseteq \mathscr{U}_1^{\perp} \cap \mathscr{U}_2^{\perp} = \{0\}$.

Assume for contradiction that $\mathscr{S}^3_{++}$ is contained in $\mathscr{W}_1^{\perp} \oplus \mathscr{W}_2^{\perp}$. This implies that

$$\mathscr{W}_1^{\perp} \oplus \mathscr{W}_2^{\perp} = \mathscr{S}^3 = \mathscr{U}_1^{\perp} \oplus \mathscr{U}_2^{\perp} \tag{10.15}$$

and thus $\mathscr{W}_k = \mathscr{U}_k$ $(k = 1,2)$. Indeed, (10.15) implies that $\dim \mathscr{W}_1^{\perp} + \dim \mathscr{W}_2^{\perp} = \dim \mathscr{U}_1^{\perp} + \dim \mathscr{U}_2^{\perp}$ which combined with the fact that $\mathscr{W}_k^{\perp} \subseteq \mathscr{U}_k^{\perp}$ $(k = 1,2)$ gives that $\dim \mathscr{W}_k^{\perp} = \dim \mathscr{U}_k^{\perp}$ $(k = 1,2)$. Lastly, using the fact that $\mathscr{U}_k \subseteq W_k$ $(k = 1,2)$ the claim follows. In turn this implies that $\mathscr{W}_1 \cap \mathscr{W}_2 = \mathscr{U}_1 \cap \mathscr{U}_2 = \{0\}$.

As $\dim \mathscr{U}_k = 3$, we have $\dim \langle u_i : i \in V_k \rangle \geq 2$ for $k = 1,2$. Say, $\{u_1, u_2\}$ and $\{u_4, u_5\}$ are linearly independent. As $\dim \langle u_i : i \in [6] \rangle = 3$, there exists a nonzero vector $\lambda \in \mathbb{R}^4$ such that $0 \neq w = \lambda_1 u_1 + \lambda_2 u_2 = \lambda_3 u_4 + \lambda_4 u_5$. Notice that the scalar w is nonzero for otherwise the vectors u_1, u_2 would be dependent. Hence we obtain that $ww^{\mathsf{T}} \in \mathscr{W}_1 \cap \mathscr{W}_2$, contradicting the fact that $\mathscr{W}_1 \cap \mathscr{W}_2 = \{0\}$.

Hence we have shown that $\mathscr{S}^3_{++} \not\subseteq \mathscr{W}_1^{\perp} \oplus \mathscr{W}_2^{\perp}$. So there exists a positive definite matrix Y which does not belong to $\mathscr{W}_1^{\perp} \oplus \mathscr{W}_2^{\perp}$. Write $Y = Y_1 + Y_2$, where $Y_k \in \mathscr{U}_k^{\perp}$ for $k = 1,2$. We may assume, say, that $Y_1 \notin \mathscr{W}_1^{\perp}$. Thus Y_1, Y_2 satisfy the lemma. $\square$

We now conclude with the main result of this section. Recall that two matrices in the relative interior of a face F of $\mathscr{E}_n$ have the same rank, while if X is in the relative interior of F and Y lies on the boundary of F then $\operatorname{rank} X > \operatorname{rank} Y$ (this follows directly from (3.7)).

10.1.26 Theorem. *For the graph $K_{3,3}$ we have that* $\operatorname{egd}(K_{3,3}) = 2$*, i.e., for any partial matrix $x \in \operatorname{ext} \mathscr{E}(K_{3,3})$ there exists a completion $X \in \operatorname{fib}(x)$ with* $\operatorname{rank} X \leq 2$.

Proof. From Lemma 10.1.23 we have that $\operatorname{rank} X \leq 3$ for every $X \in \operatorname{fib}(x)$. Assume for contradiction that there exists a partial matrix $x \in \operatorname{ext} \mathscr{E}(K_{3,3})$ with the property that $\operatorname{rank} X = 3$ for every $X \in \operatorname{fib}(x)$. We now show that this in fact implies that $\operatorname{fib}(x)$ is a singleton. Indeed, assume that $\operatorname{fib}(x)$ is not a singleton and let $X \in \operatorname{ext} \operatorname{fib}(x)$ (notice that $\operatorname{fib}(x)$ has an extreme point since it is closed convex and does not contain straight lines). Since $\operatorname{fib}(x)$ is not a singleton there exists a matrix $X' \in \operatorname{relint} \operatorname{fib}(x)$ such that $\operatorname{fib}(x) = F_{\mathscr{E}_n}(X')$. Since X is a boundary point of $\operatorname{fib}(x)$ (it is an extreme point) it follows that $X \neq X'$ and thus $\operatorname{rank} X' > \operatorname{rank} X = 3$. This contradicts Lemma 10.1.23.

On the other hand consider a matrix $X \in \operatorname{ext} \operatorname{fib}(K_{3,3})$ with $\operatorname{rank} X = 3$ and let $\{u_1, \ldots, u_6\}$ be its Gram representation. Let Y_1 and Y_2 be the matrices provided by Lemma 10.1.25 and define the matrix $Z \in \mathscr{S}^6$ by $Z_{ij} = \langle Y_k, U_{ij} \rangle$ for $i, j \in V_k$, $k \in \{1, 2\}$, and $Z_{ij} = 0$ for $i \in V_1$, $j \in V_2$. By Lemma 10.1.25, Z is a nonzero matrix with zero diagonal entries and zeros at the positions corresponding to the edges of $K_{3,3}$.

Next we show that $X + tZ \succeq 0$ for some $t > 0$, using Lemma 10.1.24. For this it is enough to verify that (10.14) holds. Assume $Xz = 0$, i.e., $a := \sum_{i \in V_1} z_i u_i = -\sum_{j \in V_2} z_j u_j$. Then,

$$z^{\mathsf{T}} Z z = \sum_{k=1,2} \sum_{i,j \in V_k} z_i z_j \langle Y_k, U_{ij} \rangle = \langle Y_1 + Y_2, a a^{\mathsf{T}} \rangle \geq 0,$$

since $Y_1 + Y_2 \succ 0$. Moreover, $z^{\mathsf{T}} Z z = 0$ implies $a = 0$ and thus $Zz = 0$ since, for $i \in V_k$, $(Zz)_i = \sum_{j \in V_k} \langle Y_k, U_{ij} \rangle z_j = \pm \langle Y_k, (u_i a^{\mathsf{T}} + a u_i^{\mathsf{T}})/2 \rangle$.

Hence, the matrix $X' = X + tZ$ is positive semidefinite and since Z has zero diagonal it is also an element of $\mathscr{E}_6$. Moreover, since the matrix Z is zero on entries corresponding to edges of $K_{3,3}$ it follows that $X' \in \operatorname{fib}(x)$. Lastly, since Z has at least one nonzero entry it follows that $X' \neq X$. This contradicts the fact that $\operatorname{fib}(x)$ is a singleton. □

We know that both graphs $K_{3,3}$ and K_5 belong to the class $\mathscr{G}_2$. We now show that they are in some sense maximal for this property.

10.1.27 Lemma. *Let G be a 2-connected graph that contains K_5 or $K_{3,3}$ as a proper subgraph. Then, G contains H_3 as a minor and thus* $\operatorname{egd}(G) \geq 3$.

Proof. The proof is based on the following observations. If G is a 2-connected graph containing K_5 or $K_{3,3}$ as a proper subgraph, then G has a minor H which is one of the following graphs: (a) H is K_5 with one more node adjacent to two nodes of K_5, (b) H is $K_{3,3}$ with one more edge added, (c) H is $K_{3,3}$ with one more node adjacent to two adjacent nodes of $K_{3,3}$. Then H contains a H_3 subgraph in cases (a) and (b), and a H_3 minor in case (c) (easy verification). Hence, $\operatorname{egd}(G) \geq \operatorname{egd}(H_3) = 3$. □

We conclude this section with a lemma that will be used in the proof of Theorem 10.2.4.

10.1.28 Lemma. *Let G be a 2-connected graph with $n \geq 6$ nodes. Then,*

(i) If G has no F_3-minor then $\omega(G) \leq 4$.

(ii) If G is chordal and has no F_3-subgraph then $\omega(G) \leq 4$.

Proof. Assume for contradiction that $\omega(G) \geq 5$ and let $U \subseteq V$ with $G[U] = K_5$. Since G is 2-connected and $n \geq 6$, there exists a node $u \notin U$ which is connected by two vertex disjoint paths to two distinct nodes $v, w \in U$, and let P_{uv} and P_{uw} be the shortest such paths. In case (i), contract the paths P_{uv} and P_{uw} to get a node adjacent to both v and w. Then, we can easily see that G has an F_3-minor, a contradiction. In case (ii), let $v' \in P_{uv}$ and $w' \in P_{uw}$ with $(v, v'), (w, w') \in E(G)$. Since G is chordal and the paths are the shortest possible, at least one of the edges (v, w') or (w, v') will be present in G. This implies that G contains an F_3 subgraph, a contradiction. $\square$

10.2 Graphs with extreme Gram dimension at most 2

In this section we characterize the class $\mathcal{G}_2$ of graphs with extreme Gram dimension at most 2. Our main result is the following:

10.2.1 Theorem. *For any graph G,*

$$\text{egd}(G) \leq 2 \textit{ if and only if } G \textit{ has no minors } F_3, H_3.$$

The graphs F_3 and H_3 are illustrated in Figure 10.4 below.

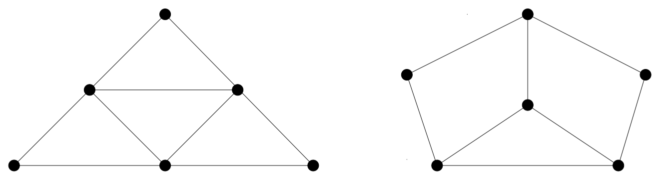

Figure 10.4: The graphs F_3 and H_3

In the previous sections we established that the graphs F_3 and H_3 are minimal forbidden minors for the class of graphs satisfying $\text{egd}(G) \leq 2$. In order to prove Theorem 10.2.1 it remains to show that a graph G having no F_3 and H_3 minors satisfies $\text{egd}(G) \leq 2$.

By Lemma 10.1.4 we may assume that G is 2-connected. Moreover, we may assume that $|V(G)| \geq 6$, since for graphs on less than 5 nodes we know that $\text{egd}(G) \leq \text{egd}(K_5) = 2$ (recall (10.9)). Additionally, since $\text{egd}(K_{3,3}) = 2$ (recall Theorem 10.1.26) we may also assume that $G \neq K_{3,3}$.

Consequently, it suffices to consider 2-connected graphs with at least 6 nodes that are different from $K_{3,3}$. Then, necessity in Theorem (10.2.1) follows from the equivalence of the first two items in the next theorem.

10.2.2 Theorem. *Let G be a 2-connected graph with $n \geq 6$ nodes and $G \neq K_{3,3}$. Then, the following assertions are equivalent.*

(i) $\text{egd}(G) \leq 2$.

(ii) *G has no minors F_3 or H_3.*

(iii) $\mathrm{la}_{\boxtimes}(G) \leq 2$*, i.e., G is a minor of $T \boxtimes K_2$ for some tree T.*

The implication $(i) \Longrightarrow (ii)$ follows from Theorem 10.1.15 and Theorem 10.1.21. Moreover, the implication $(iii) \Longrightarrow (i)$ follows from Theorem 10.1.12. The rest of this chapter is dedicated to proving the implication $(ii) \Longrightarrow (iii)$. The proof consists of two steps. First we consider the chordal case and show:

(1) **The chordal case:** Let G be a 2-connected chordal graph with $n \geq 6$ nodes. Then, G has no F_3 or H_3-minors if and only if G is a contraction minor of $T \boxtimes K_2$, for some tree T (Section 10.2.1, Theorem 10.2.4).

Then, we reduce the general case to the chordal case and show:

(2) **Reduction to the chordal case:** Let G be a 2-connected graph with $n \geq 6$ nodes and $G \neq K_{3,3}$. If G has no F_3 or H_3-minors then G is a subgraph of a chordal graph with no F_3 or H_3-minors.

Notice that in case (1), G is by assumption chordal and thus the case $G = K_{3,3}$ is automatically excluded. For case (2), we first need to exclude K_4 instead of H_3 (Section 10.2.2, Theorem 10.2.7) and then we derive from this special case the general result (Section 10.2.3, Theorem 10.2.12).

10.2.1 The chordal case

Our goal in this section is to characterize the 2-connected chordal graphs G with $\mathrm{egd}(G) \leq 2$. By Lemma 10.1.27, if $G \neq K_5$ has $\mathrm{egd}(G) \leq 2$, then $\omega(G) \leq 4$. Throughout this section we denote by $\mathscr{C}$ the family of all 2-connected chordal graphs with $\omega(G) \leq 4$. Any graph $G \in \mathscr{C}$ is a clique 2- or 3-sum of K_3's and K_4's. Note that F_3 belongs to $\mathscr{C}$ and has $\mathrm{egd}(F_3) = 3$. On the other hand, any graph $G = T \boxtimes K_2$ where T is a tree, belongs to $\mathscr{C}$ and has $\mathrm{egd}(G) = 2$. These graphs are "special clique 2-sums" of K_4's, as they satisfy the following property: every 4-clique has at most two edges which are cutsets and these two edges are not adjacent. This motivates the following definitions, useful in the proof of Theorem 10.2.4 below.

10.2.3 Definition. *Let G be a 2-connected chordal graph with $\omega(G) \leq 4$.*

(i) *An edge of G is called* free *if it belongs to exactly one maximal clique and* non-free *otherwise.*

(ii) *A 3-clique in G is called* free *if it contains at least one free edge.*

(iii) *A 4-clique in G is called* free *if it does not have two adjacent non-free edges. A free 4-clique can be partitioned as $\{a, b\} \cup \{c, d\}$, called its* two sides*, where only $\{a, b\}$ and $\{c, d\}$ can be non-free.*

(iv) *G is called free if all its maximal cliques are free.*

For instance, F_3, $K_5 \setminus e$ (the clique 3-sum of two K_4's) are not free. Hence any free graph in $\mathscr{C}$ is a clique 2-sum of free K_3's and free K_4's. Note also that $\mathrm{la}_{\boxtimes}(K_5 \setminus e) = 3$. We now show that for a graph $G \in \mathscr{C}$ the property of being free is equivalent to having $\mathrm{la}_{\boxtimes}(G) \leq 2$ and also to having $\mathrm{egd}(G) \leq 2$.

10.2.4 Theorem. *Let G be a 2-connected chordal graph with $n \geq 6$ nodes. The following assertions are equivalent:*

(i) *G has no minors F_3 or H_3.*

(ii) *G does not contain F_3 as a subgraph.*

(iii) *$\omega(G) \leq 4$ and G is free.*

(iv) *G is a contraction minor of $T \boxtimes K_2$ for some tree T.*

(v) $\mathrm{egd}(G) \leq 2$.

Proof. The implication $(i) \Rightarrow (ii)$ is clear and the implications $(iv) \Rightarrow (v) \Rightarrow (i)$ follow from earlier results.
$(ii) \Longrightarrow (iii)$: Assume that (ii) holds. By Lemma 10.1.28 (ii) it follows that $\omega(G) \leq 4$. Our first goal is to show that G does not contain clique 3-sums of K_4's, i.e., it does not contain a $K_5 \setminus e$ subgraph. For this, assume that $G[U] = K_5 \setminus e$ for some $U \subseteq V(G)$. As $|V(G)| \geq 6$ and G is 2-connected chordal, there exists a node $u \notin U$ which is adjacent to two adjacent nodes of U. Then, one can find a F_3 subgraph in G, a contradiction. Therefore, G is a clique 2-sum of K_3's and K_4's. We now show that each of them is free.

Suppose first that $C = \{a, b, c\}$ is a maximal 3-clique which is not free. Then, there exist nodes $u, v, w \notin C$ such that $\{a, b, u\}$, $\{a, c, v\}$, $\{b, c, w\}$ are cliques in G. Moreover, u, v, w are pairwise distinct (if $u = v$ then $C \cup \{u\}$ is a clique, contradicting maximality of C) and we find a F_3 subgraph in G.

Suppose now that $C = \{a, b, c, d\}$ is a 4-clique which is not free and, say, both edges $\{a, b\}$ and $\{a, c\}$ are non-free. Then, there exist nodes $u, v \notin C$ such that $\{a, b, u\}$ and $\{a, c, v\}$ are cliques. Moreover, $u \neq v$ (else we find a $K_5 \setminus e$ subgraph) and thus we find a F_3 subgraph in G. Thus (iii) holds.

$(iii) \Longrightarrow (iv)$: Assume that G is free, $G \neq K_4, K_3$ (else we are done). When all maximal cliques are 4-cliques, it is easy to show using induction on $|V(G)|$ that $G = T \boxtimes K_2$, where T is a tree and each side of a 4-clique of G corresponds to a node of T.

Assume now that G has a maximal 3-clique $C = \{a, b, c\}$. Say, $\{b, c\}$ is free and $\{a, b\}$ is a cutset. Write $V(G) = V' \cup V'' \cup \{a, b\}$, where V'' is the (vertex set of the) component of $G \setminus \{a, b\}$ containing c, and V' is the union of the other components. Now replace node a by two new nodes a', a'' and replace C by the 4-clique $C' = \{a', a'', b, c\}$. Moreover, replace each edge $\{u, a\}$ by $\{u, a'\}$ if $u \in V'$ and by $\{u, a''\}$ if $u \in V''$. Let G' be the graph obtained in this way. Then $G' \in \mathscr{C}$ is free, G' has one less maximal 3-clique than G, and $G = G'/\{a', a''\}$. Iterating, we obtain a graph $\widehat{G}$ which is a clique 2-sum of free K_4's and contains G as a contraction minor. By the above, $\widehat{G} = T \boxtimes K_2$ and thus G is a contraction minor of $T \boxtimes K_2$. $\square$

10.2.2 Structure of the graphs with no F_3 and K_4-minor

In this section we investigate the structure of the graphs with no F_3 or K_4-minors. We start with two technical lemmas.

10.2.5 Lemma. *Let G and M be two 2-connected graphs, let $\{x, y\} \notin E(G)$ be a cutset in G, and let $r \geq 2$ be the number of components of $G \setminus \{x, y\}$.*

(i) Assume that $G \in \mathscr{F}(M)$, but the graph $G + \{x, y\}$ has a M-minor with M-partition $\{V_i : i \in V(M)\}$. If $x \in V_i$ and $y \in V_j$, then $M \setminus \{i, j\}$ has at least $r \geq 2$ components (and thus $i \neq j$).

(ii) Assume that M does not have two adjacent nodes forming a cutset in M. If $G \in \mathscr{F}(M)$, then $G + \{x, y\} \in \mathscr{F}(M)$.

Proof. (i) Let $C_1, \ldots, C_r \subseteq V(G)$ be the node sets of the components of $G \setminus \{x, y\}$. As G is 2-connected, there is an $x - y$ path P_s in $G[C_s \cup \{x, y\}]$ for each $s \in [r]$. Notice that $P_s \neq \{x, y\}$ since P_s is a path in G. Our first goal is to show that every component of $M \setminus \{i, j\}$ corresponds to exactly one component of $G \setminus \{x, y\}$. For this, let U be a component of $M \setminus \{i, j\}$. By the definition of the M-partition, the graph $G[\bigcup_{k \in U} V_k]$ is connected. As $x, y \notin \bigcup_{k \in U} V_k$, we deduce that $\bigcup_{k \in U} V_k \subseteq C_s$ for some $s \in [r]$. We can now conclude the proof. Assume for contradiction that $M \setminus \{i, j\}$ has less than r components. Then there is at least one component C_s which does not correspond to any component of $M \setminus \{i, j\}$ which means that $(\bigcup_{k \neq i,j} V_k) \cap C_s = \emptyset$. Indeed, if $V_k \cap C_s$ for some $k \neq i, j$ then since C_s is a connected component of $G \setminus \{x, y\}$ it follows that $\cup_{\lambda \in U} V_\lambda \subseteq C_s$, where U is the component of $M \setminus \{i, j\}$ that contains k. Summarizing we know that $C_s \subseteq V_i \cup V_j$. Hence the path P_s is contained in $G[V_i \cup V_j]$, thus $\{V_i : i \in V(M)\}$ remains an M-partition of G (recall that $P_s \neq \{x, y\}$) and we find a M-minor in G, a contradiction. Therefore, $M \setminus \{i, j\}$ has at least $r \geq 2$ components. This implies that $\{i, j\}$ is a cutset of M since M is 2-connected it follows that $i \neq j$.

(ii) Assume $G + \{x, y\}$ has a M-minor, with corresponding M-partition $\{V_i : i \in V(M)\}$. By (i), the nodes x and y belong to two distinct classes V_i and V_j and $\{i, j\}$ is a cutset in M. By the hypothesis, this implies that $\{i, j\} \notin E(M)$ and thus M is a minor of G, a contradiction. □

We continue with a lemma that will be essential for the next theorem.

10.2.6 Lemma. *Let $G \in \mathscr{F}(K_4)$ be a 2-connected graph and let $\{x, y\} \notin E(G)$. If there are at least three (internally vertex) disjoint paths from x to y, then $\{x, y\}$ is a cutset and $G \setminus \{x, y\}$ has at least 3 components.*

Proof. Let P_1, P_2, P_3 be distinct vertex disjoint paths from x to y. Then $P_1 \setminus \{x, y\}$, $P_2 \setminus \{x, y\}$ and $P_3 \setminus \{x, y\}$ lie in distinct components of $G \setminus \{x, y\}$, for otherwise G would contain a homeomorph of K_4. □

We now arrive at the main result of this section.

10.2.7 Theorem. *Let $G \in \mathscr{F}(F_3, K_4)$ be a 2-connected graph on $n \geq 6$ nodes. Then, there exists a chordal graph $Q \in \mathscr{F}(F_3, K_4)$ containing G as a subgraph.*

Proof. Let G be a 2-connected graph in $\mathscr{F}(F_3, K_4)$. As a first step, consider $\{x, y\} \notin E(G)$ such that there exist at least three disjoint paths in G from x to y. Then, Lemma 10.2.6 implies that $\{x, y\}$ is a cutset of G and $G \setminus \{x, y\}$ has at least three components. As a first step we show that we can add the edge $\{x, y\}$ without creating a K_4 or F_3-minor, i.e., $G + \{x, y\} \in \mathscr{F}(F_3, K_4)$.

As $\{x, y\}$ is a cutset, Lemma 10.2.5 (ii) applied for $M = K_4$ gives that $G + \{x, y\}$ does not have a K_4 minor. Assume for contradiction that $G + \{x, y\}$ has an F_3 minor. Again, Lemma 10.2.5 (i) applied for $M = F_3$ implies that $x \in V_i, y \in V_j$, where $F_3 \setminus \{i, j\}$ has at least 3 components. Clearly there is no such pair of vertices in F_3

so we arrived at a contradiction. Consequently, we can add edges iteratively until we obtain a graph $\widehat{G} \in \mathscr{F}(F_3, K_4)$ containing G as a subgraph and satisfying:

$$\forall \{x,y\} \notin E(\widehat{G}) \text{ there are at most two disjoint } x-y \text{ paths in } \widehat{G}. \tag{10.16}$$

If $\widehat{G}$ is chordal we are done. So consider a chordless circuit C in $\widehat{G}$. Note that any circuit C' distinct from C, which meets C in at least two nodes, meets C in exactly two nodes (if they meet in at least 3 nodes then we can find a F_3 minor) that are adjacent (if they are not adjacent then there exist three internally vertex disjoint paths between them, contradicting (10.16)). Call an edge of C *busy* if it is contained in some circuit $C' \neq C$. If $e_1 \neq e_2$ are two busy edges of C and $C_i \neq C$ is a circuit containing e_i, then C_1, C_2 are (internally) disjoint (use (10.16)). Hence C can have at most two busy edges, for otherwise one would find a F_3-minor in $\widehat{G}$.

We now show how to triangulate C without creating a K_4 or F_3-minor: If C has two busy edges denoted, say, $\{v_1, v_2\}$ and $\{v_k, v_{k+1}\}$ (possibly $k = 2$), then we add the edges $\{v_1, v_i\}$ for $i \in \{3, \dots, k\}$ and the edges $\{v_k, v_i\}$ for $i \in \{k+2, \dots, |C|\}$, see Figure 10.5 a). If C has only one busy edge $\{v_1, v_2\}$, add the edges $\{v_1, v_i\}$ for $i \in \{3, \dots, |C|-1\}$, see Figure 10.5 b). (If C has no busy edge then $G = C$, triangulate from any node and we are done).

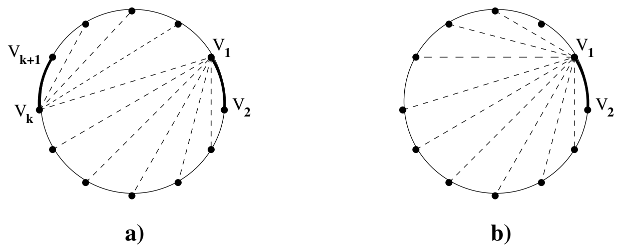

Figure 10.5: Triangulating a chordless circuit with a) two or b) one busy edge.

Let Q denote the graph obtained from $\widehat{G}$ by triangulating all its chordless circuits in this way. Hence Q is a chordal extension of $\widehat{G}$ (and thus of G). We show that $Q \in \mathscr{F}(F_3, K_4)$. First we see that $Q \in \mathscr{F}(K_4)$ by applying iteratively Lemma 10.2.5 (ii) (for $M = K_4$): For each $i \in \{3, \dots, k\}$, $\{v_1, v_i\}$ is a cutset of $\widehat{G}$ and of $\widehat{G} + \{\{v_1, v_j\} : j \in \{3, \dots, i-1\}\}$ (and analogously for the other added edges $\{v_k, v_i\}$). Hence Q is a clique 2-sum of triangles. We now verify that each triangle is free which will conclude the proof, using Theorem 10.2.4.

For this let $\{a,b,c\}$ be a triangle in Q. First note that if (say) $\{a,b\} \in E(Q) \setminus E(\widehat{G})$, then a,b,c lie on a common chordless circuit C of $\widehat{G}$. Indeed, let C be a chordless circuit of $\widehat{G}$ containing a,b and assume $c \notin C$. By (10.16), $\widehat{G} \setminus \{a,b\}$ has at most two components and thus there is a path from c to one of the two paths composing $C \setminus \{a,b\}$. Together with the triangle $\{a,b,c\}$ this gives a homeomorph of K_4 in Q, contradicting $Q \in \mathscr{F}(K_4)$, just shown above. Hence the triangle $\{a,b,c\}$ lies in C and thus has a free edge.

Suppose now that $\{a,b,c\}$ is a triangle contained in $\widehat{G}$. If it is not free then there is a F_3 on $\{a,b,c,x,y,z\}$ where x (resp., y, and z) is adjacent to a,b (resp., a,c, and b,c). Say $\{x,a\} \in E(Q) \setminus E(\widehat{G})$ (as there is no F_3 in $\widehat{G}$). Then x,a,b lie on a chordless circuit C of $\widehat{G}$ and $\{x,b\} \in E(\widehat{G})$ (since $\{a,b\}$ is a busy edge). Moreover, $c,y,z \notin C$ for otherwise we get a K_4-minor in Q. Then delete the edge

$\{x, a\}$ and replace it by the $\{x, a\}$-path along C. Do the same for any other edge of $E(Q) \setminus E(\widehat{G})$ connecting y and z to $\{a, b, c\}$. After that we get a F_3-minor in $\widehat{G}$, a contradiction. $\square$

10.2.3 Structure of the graphs with no F_3 and H_3-minor

Here we investigate the graphs $G \in \mathscr{F}(F_3, H_3)$. By the results in Section 10.2.2 we may assume that G contains some homeomorph of K_4. Figure 10.6 shows a homeomorph of K_4, where the original nodes are denoted as 1,2,3,4 and called its *corners*, and the wiggled lines correspond to subdivided edges (i.e., to paths P_{ij} between the corners $i \neq j \in [4]$).

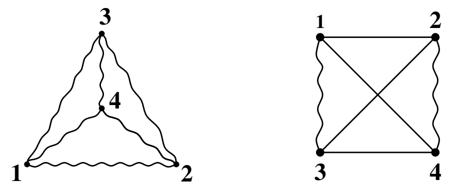

Figure 10.6: A homeomorph of K_4 and its two sides (cf. Lemma 10.2.8)

To help the reader visualize F_3 and H_3 we use Figure 10.7. Notice the special role of node 5 in H_3 (denoted by a square) and of the (dashed) triangle $\{1, 2, 3\}$.

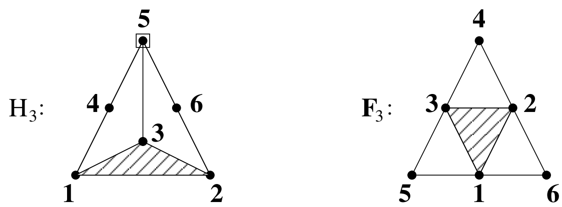

Figure 10.7: The graphs H_3 and F_3.

The starting point of the proof is to investigate the structure of homeomorphs of K_4 in a graph of $\mathscr{F}(H_3)$.

10.2.8 Lemma. *Let G be a 2-connected graph in $\mathscr{F}(H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. Then there is a partition of the corner nodes of H into $\{1, 3\}$ and $\{2, 4\}$ for which the following holds.*

(i) Only the paths P_{13} and P_{24} can have more than 2 nodes.

(ii) Every component of $G \setminus H$ is connected to P_{13} or to P_{24}.

Then P_{13} and P_{24} are called the two sides *of H (cf. Figure 10.6).*

Proof. We use the graphs from Figure 10.8 which all contain a subgraph H_3.
Case 1: $H = K_4$. If $G \setminus H$ has a unique component C then $|C| \geq 2$ as $n \geq 6$. If C is connected to two nodes of H, then the conclusion of the lemma holds. Otherwise,

C is connected to at least three nodes in H and then the graph from Figure 10.8 a) is a minor of G, a contradiction.

If there are at least two components in $G \setminus H$, then they cannot be connected to two adjacent edges of H for, otherwise, the graph of Figure 10.8 b) is a minor of G, a contradiction. Hence the lemma holds.

Case 2: $H \neq K_4$. Say, P_{13} has at least 3 nodes. Then the edges $\{1, i\}$, $\{3, i\}$ $(i = 2, 4)$ cannot be subdivided (else H is a homeomorph of H_3). So (i) holds. We now show (ii). Indeed, if a component of $G \setminus H$ is connected to both P_{13} and P_{24}, then at least one of the graphs in Figure 10.8 c) and d) will be a minor of G, a contradiction. □

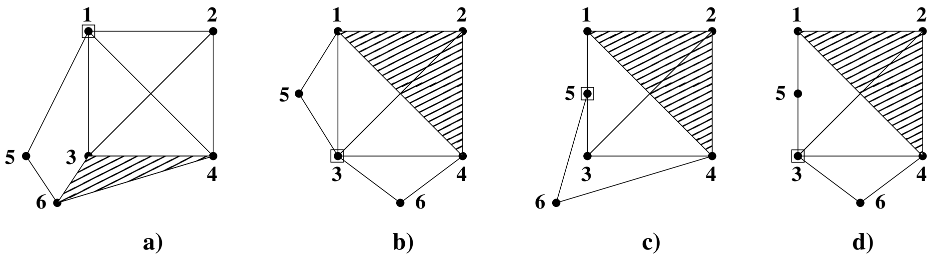

Figure 10.8: Bad subgraphs in the proof of Lemma 10.2.8.

Lemma 10.2.8 implies that there is no path with at least 3 nodes between the sides of a K_4-homeomorph. We now show that, moreover, there is no additional edge between the two sides. More precisely:

10.2.9 Lemma. *Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathscr{F}(H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. Then there exists no edge between the two sides of H except between their endpoints.*

Proof. Say, P_{13} and P_{24} are the two sides of H. Assume for a contradiction that $\{a, b\} \in E(G)$, where a lies on P_{13} and b on P_{24}.

Assume first that a is an internal node of P_{13} and b is an internal node of P_{24}. If $|V(H)| = 6$, then $H = K_{3,3}$ and Lemma 10.1.27 implies that G has a H_3 minor, a contradiction. Hence, $|V(H)| > 6$ and we can assume w.l.o.g. that the path from 1 to a within P_{13} has at least 3 nodes. Then G contains a homeomorph of K_4 with corner nodes $1, b, 4, a$, where the two paths from 1 to a and from 1 to b (via 2) have at least 3 nodes, giving a H_3 minor and thus a contradiction.

Assume now that only a is an internal node of P_{13} and, say $b = 2$. If $|V(H)| = 5$, then $G \setminus H$ has at least one component. By Lemma 10.2.8, this component connects either to the path P_{13} or to the edge $\{2, 4\}$. In both cases, it is easy to verify that one of the graphs in Figure 10.9 will be a minor of G, a contradiction since all of them have a H_3 subgraph. On the other hand, if $|V(H)| \geq 6$, then one of the paths from 1 to a, from a to 3 (within P_{13}), or P_{24} is subdivided. This implies that G contains a homeomorph of K_4 with corner nodes $a, 1, 2, 4$ or $a, 2, 3, 4$, which thus contains two adjacent subdivided edges, giving a H_3 minor. □

Lemmas 10.2.8 and 10.2.9 imply directly:

10.2.10 Corollary. *Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathscr{F}(H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. Then the endnodes of at least one of*

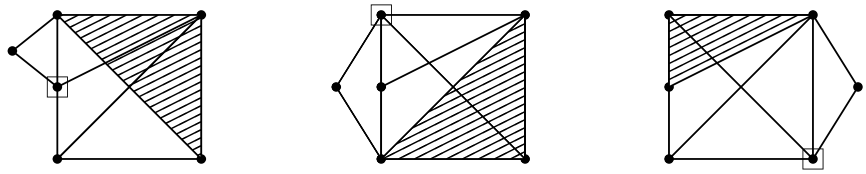

Figure 10.9: Bad subgraphs in the proof of Lemma 10.2.9.

the two sides of H form a cutset in G. Moreover, if P_{13} is a side of H and its endnodes $\{1,3\}$ do not form a cutset, then $P_{13} = \{1,3\}$ and there is no component of $G \setminus H$ which is connected to P_{13}.

We now show that one may add edges to G so that all minimal homeomorphs of K_4 are 4-cliques, without creating a F_3 or H_3 minor.

10.2.11 Lemma. *Let $G \neq K_{3,3}$ be a 2-connected graph in $\mathscr{F}(F_3, H_3)$ on $n \geq 6$ nodes and let H be a homeomorph of K_4 contained in G. The graph obtained by adding to G the edges between the endpoints of the sides of H belongs to $\mathscr{F}(F_3, H_3)$.*

Proof. Say P_{13} and P_{24} are the sides of H. Assume $|V(P_{13})| \geq 3$ and $\{1,3\} \notin E(G)$. By Corollary 10.2.10, $\{1,3\}$ is a cutset in G. We show that $\widehat{G} = G + \{1,3\} \in \mathscr{F}(F_3, H_3)$. First, applying Lemma 10.2.5 (ii) with $M = H_3$ and $\{x, y\} = \{1,3\}$, we obtain that $\widehat{G} \in \mathscr{F}(H_3)$.

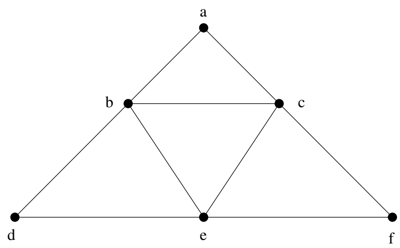

Figure 10.10: A labelling of F_3 used in Lemma 10.2.11.

Next, assume for contradiction that $\widehat{G}$ has a F_3 minor, where the labelling of F_3 is given in Figure 10.10. Applying Lemma 10.2.5 (i) with $M = F_3$ and $\{x, y\} = \{1,3\}$, we see that the nodes 1 and 3 belong to distinct classes of the F_3-partition, which corresponds to a cutset of F_3. Say, $1 \in V_e$ and $3 \in V_c$. Then the nodes 2 and 4 do not lie in $V_e \cup V_c$ (for otherwise, one would have an F_3-partition in G). Next we show that the nodes 2 and 4 do not belong to the same class of the F_3-partition. Assume for contradiction that $2, 4 \in V_k$. If $\{2,4\}$ is not a cutset in G then, by Corollary 10.2.10, $P_{24} = \{2,4\}$ and no component of $G \setminus H$ connects to $\{2,4\}$. Hence $V_k = \{2,4\}$ and we can move node 2 to the class V_e, so that we obtain a F_3-partition of G, a contradiction. If $\{2,4\}$ is a cutset of G, then every component of $G \setminus \{2,4\}$ except the one containing 1 and 3 has to lie within V_k, so we can again move node 2 to V_e and obtain a F_3-partition of G.

Accordingly, the nodes $1, 2, 3$ and 4 belong to distinct classes and we can assume without loss of generality that $2 \notin V_f$. Observe that every $1-2$ path in G is either the edge $\{1,2\}$ or meets the nodes 3 or 4. Similarly, every $2-3$ path in G is either the edge $\{2,3\}$ or meets the nodes 1 or 4. An easy case analysis shows that whatever the position of nodes 2 and 4 in the F_3-partition we always find a $1-2$ or a $2-3$ path violating the above conditions. □

We are now ready to show the main result of this section.

10.2.12 Theorem. *Let G be a 2-connected graph with $n \geq 6$ nodes and $G \neq K_{3,3}$. If $G \in \mathscr{F}(F_3, H_3)$ then there exists a chordal graph $Q \in \mathscr{F}(F_3, H_3)$ containing G as a subgraph.*

Proof. If $G \in \mathscr{F}(F_3, K_4)$ then we are done by Theorem 10.2.7. Otherwise, we augment the graph G by adding the edges between the endpoints of the sides of every homeomorph of K_4 contained in G. Let $\widehat{G}$ be the graph obtained in this way. By Lemma 10.2.11, we know that $\widehat{G} \in \mathscr{F}(F_3, H_3)$. Hence, for each K_4-homeomorph H in $\widehat{G}$, its corners form a 4-clique. Moreover, if C, C' are two distinct 4-cliques of $\widehat{G}$, then $C \cap C'$ is contained in a side of C and C'.

Consider a 4-clique $C = \{1,2,3,4\}$ in $\widehat{G}$, say with sides $\{1,3\}$, $\{2,4\}$ (so each component of $\widehat{G} \setminus C$ connects to $\{1,3\}$ or to $\{2,4\}$, by Lemma 10.2.8). Pick an edge f between the two sides (i.e., $f = \{i, j\}$ with $i \in \{1,3\}$, $j \in \{2,4\}$) and delete this edge f from $\widehat{G}$. We repeat this process with every 4-clique in $\widehat{G}$ and obtain the graph $G_0 = \widehat{G} \setminus \{f_1, \ldots, f_k\}$, if $\widehat{G}$ has k 4-cliques.

By construction, G_0 belongs to $\mathscr{F}(F_3, K_4)$ and is 2-connected. Hence, we can apply Theorem 10.2.7 to G_0 and obtain a chordal graph $Q_0 \in \mathscr{F}(F_3, K_4)$ containing G_0 as a subgraph. Hence, Q_0 is a clique 2-sum of free triangles. It suffices now to show that the augmented graph $Q = Q_0 + \{f_i : i \in [k]\}$ is a clique 2-sum of free K_3's and K_4's. Then Q is a chordal graph in $\mathscr{F}(F_3, H_3)$ (by Theorem 10.2.4) containing $\widehat{G}$ and thus G, and the proof is completed.

For this, consider again a 4-clique $C = \{1,2,3,4\}$ in $\widehat{G}$ with sides $\{1,3\}$ and $\{2,4\}$. Then, each component of $\widehat{G} \setminus C$ connects to $\{1,3\}$ or $\{2,4\}$. We claim that the same holds for each component of $Q_0 \setminus C$. Indeed, a component of $Q_0 \setminus C$ is a union of some components of $\widehat{G} \setminus C$. Thus it connects to two nodes (to 1,3, or to 2,4), or to at least three nodes of C. But the latter case cannot occur since we would then find a K_4 minor in Q_0.

Assume that the edge $f = \{1,4\}$ was deleted from the 4-clique C when making the graph G_0. We now show that adding it back to Q_0 results in a free graph. Indeed, by adding the edge $\{1,4\}$ we only replace the two maximal 3-cliques $\{1,3,4\}$ and $\{1,2,4\}$ by a new maximal 4-clique $\{1,2,3,4\}$, which is free. We iterate this process for each of the edges $f_1, \ldots, f_k$ and obtain that $Q = Q_0 + \{f_i : i \in [k]\}$ is the clique 2-sum of free K_3's and K_4's. Summarizing, Q is a 2-connected chordal graph with $\omega(G) \leq 4$ which is free. Then, Theorem 10.2.4 (iii) implies that Q does not have F_3 or H_3 as minors. □

10.3 Characterization of graphs with $\mathrm{la}_{\boxtimes}(G) \leq 2$

Recall that the *largeur d'arborescence* of a graph G, denoted by $\mathrm{la}_{\square}(G)$, is defined as the smallest integer $k \geq 1$ such that G is a minor of $T \square K_k$, for some tree T. Colin de Verdière [43] introduced the largeur d'arborescence as upper bound for his graph parameter $\nu(\cdot)$, which is defined as the maximum corank of a matrix

$A \in \mathcal{S}_+^n$ satisfying the Strong Arnold Property and moreover: $A_{ij} = 0$ if and only if $i \neq j$ and $\{i, j\} \notin E(G)$.

In [43] it was shown that $\nu(\cdot)$ is minor monotone and that for any graph G, $\nu(G) \leq \mathrm{la}_{\square}(G)$. Moreover, this holds with equality for the family of graphs G_r, i.e., $\nu(G_r) = \mathrm{la}_{\square}(G_r) = r$ for all $r \geq 2$ (recall Section 10.1.4). Furthermore,

$$\mathrm{la}_{\square}(G) \leq 1 \Longleftrightarrow \nu(G) \leq 1 \Longleftrightarrow G \text{ has no minor } K_3. \tag{10.17}$$

Lastly, Kotlov [70] shows:

$$\mathrm{la}_{\square}(G) \leq 2 \Longleftrightarrow \nu(G) \leq 2 \Longleftrightarrow G \text{ has no minors } F_3, K_4. \tag{10.18}$$

The most work in obtaining the characterization (10.18) is to show that $\mathrm{la}_{\square}(G) \leq 2$ if $G \in \mathscr{F}(K_4, F_3)$. In fact, this also follows from our characterization of the class $\mathscr{F}(K_4, F_3)$. Indeed, if $G \in \mathscr{F}(K_4, F_3)$ is 2-connected then we have shown that G is subgraph of G' which is a clique 2-sum of free triangles. Now our argument in the proof of Theorem 10.2.4 also shows that G' is a contraction minor of $T \square K_2$ for some tree T (as each triangle of G' arises as contraction of a 4-clique which can be replaced by a 4-circuit). In this sense our characterization is a refinement of Kotlov's result tailored to our needs.

We now characterize the graphs with $\mathrm{la}_{\boxtimes}(G) \leq 2$. The *wheel* W_5 is obtained from the circuit C_4 by adding a node adjacent to all nodes of C_4.

10.3.1 Theorem. *For a graph G,* $\mathrm{la}_{\boxtimes}(G) \leq 2$ *if and only if* $G \in \mathscr{F}(F_3, H_3, W_5)$.

Proof. We already know that $\mathrm{la}_{\boxtimes}(G) \geq \mathrm{egd}(G) = 3$ for $G = F_3, H_3$. Suppose for contradiction that $\mathrm{la}_{\boxtimes}(W_5) \leq 2$. Then $\mathrm{la}_{\boxtimes}(W_5) = \mathrm{la}_{\boxtimes}(H)$ where H is a chordal extension of W_5 and H is a contraction minor of some $T \boxtimes K_2$. As W_5 is not chordal, H contains W_5 with one added chord on its 4-circuit, i.e., H contains $K_5 \setminus e$ and thus $\mathrm{la}_{\boxtimes}(H) \geq \mathrm{la}_{\boxtimes}(K_5 \setminus e) = 3$. Therefore, F_3, H_3, W_5 are forbidden minors for the property $\mathrm{la}_{\boxtimes}(G) \leq 2$. Conversely, assume that $G \in \mathscr{F}(F_3, H_3, W_5)$ is 2-connected, we show that $\mathrm{la}_{\boxtimes}(G) \leq 2$. This is clear if G has $n \leq 4$ nodes, or if G has $n = 5$ nodes and it has a node of degree 2. If G has $n = 5$ nodes and each node has degree at least 3, then one can easily verify that G contains W_5. If G has $n \geq 6$ nodes then $\mathrm{la}_{\boxtimes}(G) \leq 2$ follows from Theorem 10.2.2 (since $G \neq K_{3,3}$ as $W_5 \preceq K_{3,3}$). $\square$

Summarizing, it is known that $\nu(G) \leq \mathrm{la}_{\square}(G)$ and $\mathrm{egd}(G) \leq \mathrm{la}_{\boxtimes}(G) \leq \mathrm{la}_{\square}(G)$. Moreover, by combining (10.17), (10.18) and Theorem 10.2.1 it follows that if G is a graph with $\nu(G) \leq 2$, then $\mathrm{egd}(G) = \nu(G)$. Furthermore, it is known that $\nu(K_n) = n - 1$ [43] and thus $\nu(K_n) > \mathrm{la}_{\boxtimes}(K_n) \geq \mathrm{egd}(K_n)$ if $n \geq 4$.

An interesting open question is whether the inequality $\mathrm{egd}(G) \leq \nu(G)$ holds in general. We point out that the analogous inequality $\nu^{=}(G) \leq \mathrm{gd}(G)$ was shown earlier in Section 6.2. Recall that the parameter $\nu^{=}(\cdot)$ is the analogue of $\nu(\cdot)$ studied by van der Holst [129] (same definition as $\nu(G)$, but now requiring only that $A_{ij} = 0$ for $\{i, j\} \in E(G)$ and allowing zero entries at positions on the diagonal and at edges), and $\nu^{=}(\cdot)$ satisfies: $\nu(G) \leq \nu^{=}(G)$.

11
Universally completable frameworks

In this chapter we address the following three topics: positive semidefinite matrix completions, universal rigidity of frameworks, and the Strong Arnold Property. We show some strong connections among these topics, using semidefinite programming as unifying theme. Our main contribution is a sufficient condition for constructing partial psd matrices which admit a unique completion to a full psd matrix. As we have seen already in earlier chapters, such partial matrices are an essential tool in the study of the Gram dimension and the extreme Gram dimension of a graph. Using this sufficient condition we can recover most constructions from Chapters 5 and 10. Additionally, we derive an elementary proof of Connelly's sufficient condition for universal rigidity of tensegrity frameworks and we investigate the links between these two sufficient conditions. Lastly, we also give a geometric characterization of psd matrices satisfying the Strong Arnold Property in terms of nondegeneracy of a certain semidefinite program.

The content of this chapter is based on joint work with M. Laurent [80].

11.1 Introduction

For a graph $G = (V = [n], E)$, recall that a vector $a \in \mathbb{R}^{V \cup E}$ is called a *G-partial psd matrix* if it admits at least one completion to a full psd matrix. Moreover, $\mathscr{S}_+(G)$ denotes the set of all G-partial psd matrices.

Notice that we can define G-partial psd matrices in terms of Gram representations. Namely, $a \in \mathscr{S}_+(G)$ if and only if there exist vectors $p_1, \ldots, p_n \in \mathbb{R}^d$ (for some $d \geq 1$) such that $a_{ij} = p_i^{\mathsf{T}} p_j$ for all $\{i, j\} \in V \cup E$. This leads to the notion of frameworks which will make the link between the Gram (spherical) setting of this section and the Euclidean distance setting considered in Section 11.2.

We start the discussion with some necessary definitions. A *tensegrity graph* is a graph G whose edge set is partitioned into three sets: $E = B \cup C \cup S$, whose members are called *bars, cables* and *struts*, respectively. A *tensegrity framework* $G(\mathbf{p})$ consists of a tensegrity graph G together with an assignment of vectors $\mathbf{p} = \{p_1, \ldots, p_n\}$ to the nodes of G. A *bar framework* is a tensegrity framework where $C = S = \emptyset$.

Given a tensegrity framework $G(\mathbf{p})$ consider the following pair of primal and dual semidefinite programs:

$$\begin{aligned} \sup_X \{0 : X \succeq 0, \quad & \langle E_{ij}, X\rangle = p_i^\mathsf{T} p_j \text{ for } \{i,j\} \in V \cup B, \\ & \langle E_{ij}, X\rangle \le p_i^\mathsf{T} p_j \text{ for } \{i,j\} \in C, \\ & \langle E_{ij}, X\rangle \ge p_i^\mathsf{T} p_j \text{ for } \{i,j\} \in S\} \end{aligned} \tag{$\mathscr{P}_G$}$$

and

$$\begin{aligned} \inf_{y,Z} \{\textstyle\sum_{ij \in V\cup E} y_{ij} p_i^\mathsf{T} p_j : \quad & \textstyle\sum_{ij\in V\cup E} y_{ij} E_{ij} = Z \succeq 0, \\ & y_{ij} \ge 0 \text{ for } \{i,j\} \in C, \\ & y_{ij} \le 0 \text{ for } \{i,j\} \in S\}. \end{aligned} \tag{$\mathscr{D}_G$}$$

The next definition captures the analogue of the notion of universal rigidity for the Gram setting.

11.1.1 Definition. *A tensegrity framework $G(\mathbf{p})$ is called* universally completable *if the matrix* Gram$(p_1, \ldots, p_n)$ *is the unique solution of the semidefinite program* ($\mathscr{P}_G$).

In other words, a universally completable framework $G(\mathbf{p})$ corresponds to a G-partial psd matrix $a \in \mathscr{S}_+(G)$, where $a_{ij} = p_i^\mathsf{T} p_j$ for all $\{i,j\} \in V \cup E$, that admits a unique completion to a full psd matrix. Consequently, identifying sufficient conditions guaranteeing that a framework $G(\mathbf{p})$ is universally completable will allow us to construct G-partial matrices with a unique psd completion.

11.1.1 A sufficient condition for universal completability

In this section we derive a sufficient condition for showing that a tensegrity framework is universally completable. We use the following notation: For a graph $G = (V, E)$, we denote by $\overline{E}$ the set of pairs $\{i,j\}$ with $i \neq j$ and $\{i,j\} \notin E$, corresponding to the non-edges of G.

11.1.2 Theorem. *Let $G = ([n], E)$ be a tensegrity graph with $E = B \cup C \cup S$ and consider a tensegrity framework $G(\mathbf{p})$ in $\mathbb{R}^d$ such that $p_1, \ldots, p_n$ span linearly $\mathbb{R}^d$. Assume there exists a matrix $Z \in \mathscr{S}^n$ satisfying the conditions (i)-(vi):*

(i) Z is positive semidefinite.

(ii) $Z_{ij} = 0$ for all $\{i,j\} \in \overline{E}$.

(iii) $Z_{ij} \ge 0$ for all (cables) $\{i,j\} \in C$ and $Z_{ij} \le 0$ for all (struts) $\{i,j\} \in S$.

(iv) Z has corank d.

(v) $\sum_{j\in V} Z_{ij} p_j = 0$ for all $i \in [n]$.

(vi) For any matrix $R \in \mathscr{S}^d$ the following holds:

$$p_i^\mathsf{T} R\, p_j = 0 \ \forall \{i,j\} \in V \cup B \cup \{\{i,j\} \in C \cup S : Z_{ij} \neq 0\} \Longrightarrow R = 0. \tag{11.1}$$

Then the tensegrity framework $G(\mathbf{p})$ is universally completable.

Proof. Set $X = \text{Gram}(p_1, \ldots, p_n)$. Assume that $Y \in \mathscr{S}_+^n$ is another matrix which is feasible for the program $(\mathscr{P}_G)$, say $Y = \text{Gram}(q_1, \ldots, q_n)$ for some vectors $q_1, \ldots, q_n$. Our goal is to show that $Y = X$. By (v), $ZX = 0$ and thus $\text{Ran}\, X \subseteq \text{Ker}\, Z$. Moreover, $\dim \text{Ker}\, Z = d$ by (iv), and $\text{rank} X = d$ since $\text{lin}\{p_1, \ldots, p_n\} = \mathbb{R}^d$. This implies that $\text{Ker} X = \text{Ran}\, Z$.

By (ii) we can write $Z = \sum_{\{i,j\} \in V \cup E} Z_{ij} E_{ij}$. Next notice that

$$0 \le \langle Z, Y \rangle = \langle \sum_{\{i,j\} \in V \cup E} Z_{ij} E_{ij}, Y \rangle \le \sum_{\{i,j\} \in V \cup E} Z_{ij} \langle E_{ij}, X \rangle = \langle Z, X \rangle = 0, \tag{11.2}$$

where the first (left most) inequality follows from the fact that $Y, Z \succeq 0$ and the second one from the feasibility of Y for $(\mathscr{P}_G)$ and the sign conditions (iii) on Z. This gives $\langle Z, Y \rangle = 0$, which implies that $\text{Ker}\, Y \supseteq \text{Ran}\, Z$ and thus $\text{Ker}\, Y \supseteq \text{Ker} X$.

Write $X = PP^{\mathsf{T}}$, where $P \in \mathbb{R}^{n \times d}$ has rows $p_1^{\mathsf{T}}, \ldots, p_n^{\mathsf{T}}$. From the inclusion $\text{Ker}\,(Y - X) \supseteq \text{Ker} X$, we deduce that $Y - X = PRP^{\mathsf{T}}$ for some matrix $R \in \mathscr{S}^d$.

Since equality holds throughout in (11.2), we obtain that $\langle E_{ij}, Y - X \rangle = 0$ for all $\{i, j\} \in C \cup S$ with $Z_{ij} \neq 0$. Additionally, as X, Y are both feasible for $(\mathscr{P}_G)$, we have that $\langle E_{ij}, Y - X \rangle = 0$ for all $\{i, j\} \in V \cup B$. Substituting PRP^{T} for $Y - X$, we obtain that $p_i^{\mathsf{T}} R p_j = 0$ for all $\{i, j\} \in V \cup B$ and all $\{i, j\} \in C \cup S$ with $Z_{ij} \neq 0$. We can now apply (vi) to get $R = 0$. This gives $Y = X$, which concludes the proof. □

Note that the conditions (i)-(iii) express that Z is feasible for the dual semidefinite program $(\mathscr{D}_G)$. In analogy to the Euclidean setting (see Section 11.2), such matrix Z is called a *spherical stress matrix* for the framework $G(\mathbf{p})$. Moreover, (v) says that Z is dual optimal and (iv) says that $X = \text{Gram}(p_1, \ldots, p_n)$ and Z are strictly complementary solutions to the primal and dual semidefinite programs $(\mathscr{P}_G)$ and $(\mathscr{D}_G)$. Finally, in the case of bar frameworks (when $C = S = \emptyset$), condition (vi) means that Z is dual nondegenerate. Hence, for bar frameworks, Theorem 11.1.2 also follows as a direct application of Theorem 3.3.7.

As a last remark notice that the assumptions of Theorem 11.1.2 imply that $n \ge d$. Moreover, for $n = d$, the matrix Z is the zero matrix and in this case (11.1) reads: $p_i^{\mathsf{T}} R p_j = 0$ for all $\{i, j\} \in V \cup B$ then $R = 0$. Observe that this condition can be satisfied only when $G = K_n$ and $C = S = \emptyset$, so that Theorem 11.1.2 is useful only in the case when $d \le n - 1$.

11.1.2 Applying the sufficient condition

In this section we use Theorem 11.1.2 to construct several instances of partial psd matrices admitting a unique psd completion. Such partial matrices have been a crucial ingredient in proving lower bounds for $\text{gd}(\cdot)$ and $\text{egd}(\cdot)$, in Chapters 5 and 10, respectively. While the proofs there for unicity of the psd completion consisted of ad hoc arguments and case checking, Theorem 11.1.2 provides us with a unified and systematic approach for constructing such instances.

In all examples below we only deal with bar frameworks and hence we apply Theorem 11.1.2 with $C = S = \emptyset$. In particular, there are no sign conditions on the stress matrix Z and moreover condition (11.1) assumes the simpler form: If $p_i^{\mathsf{T}} R p_j = 0$ for all $\{i, j\} \in V \cup E$ then $R = 0$.

Example 1: The octahedral graph. Consider a framework for the octahedral graph $K_{2,2,2}$ defined as follows:

$$p_1 = e_1,\ p_2 = e_2,\ p_3 = e_1 + e_2,\ p_4 = e_3,\ p_5 = e_4,\ p_6 = e_5,$$

where e_i ($i \in [5]$) denote the standard unit vectors in $\mathbb{R}^5$ and the numbering of the nodes refers to Figure 5.1. In Lemma 5.3.1 we showed that the corresponding $K_{2,2,2}$-partial matrix $a = (p_i^{\mathsf{T}} p_j) \in \mathscr{S}_+(K_{2,2,2})$ admits a unique psd completion. This result follows easily, using Theorem 11.1.2. Indeed it is easy to check that condition (11.1) holds. Moreover, the matrix $Z = (1,1,-1,0,0,0)(1,1,-1,0,0,0)^{\mathsf{T}}$ is psd with corank 5, it is supported by $K_{2,2,2}$, and satisfies $\langle Z, \mathrm{Gram}(p_1,\dots,p_5)\rangle = 0$. Hence Theorem 11.1.2 applies and the claim follows.

Example 2: The family of graphs F_r. For an integer $r \geq 2$, we define a graph $F_r = (V_r, E_r)$ with $r + \binom{r}{2}$ nodes denoted as v_i (for $i \in [r]$) and v_{ij} (for $1 \leq i < j \leq r$). It consists of a central clique of size r based on the nodes $v_1,\dots,v_r$ together with the cliques C_{ij} on the nodes $\{v_i, v_j, v_{ij}\}$. The graphs F_3 and F_4 are shown in Figure 11.1.2 below. We construct a framework in $\mathbb{R}^r$ for the graph F_r as follows:

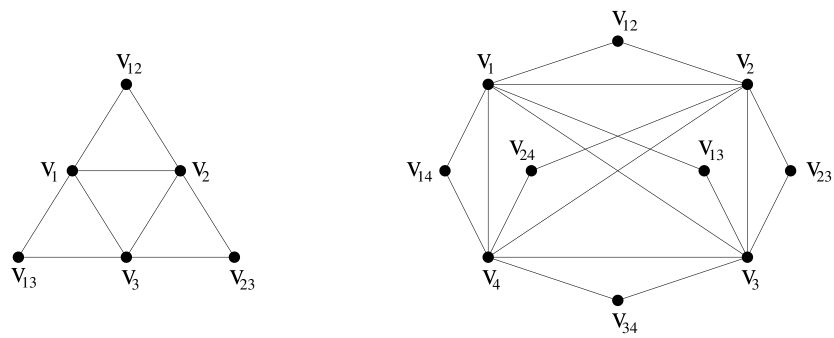

Figure 11.1: The graphs F_3 and F_4.

$$p_{v_i} = e_i \text{ for } i \in [r] \text{ and } p_{v_{ij}} = e_i + e_j \text{ for } 1 \leq i < j \leq r.$$

In Section 10.1.4 we showed that for any $r \geq 2$ the corresponding F_r-partial matrix admits a unique psd completion. We now show this result, using Theorem 11.1.2.

Fix $r \geq 2$. It is easy to check that (11.1) holds. Define the nonzero matrix $Z_r = \sum_{1\leq i<j\leq r} u_{ij} u_{ij}^{\mathsf{T}}$, where the vectors $u_{ij} \in \mathbb{R}^{r+\binom{r}{2}}$ are defined as follows: For $1 \leq k \leq r$, $(u_{ij})_k = 1$ if $k \in \{i,j\}$ and 0 otherwise; for $1 \leq k < l \leq r$, $(u_{ij})_{kl} = -1$ if $\{k,l\} = \{i,j\}$ and 0 otherwise. By construction, Z_r is psd, it is supported by the graph F_r, $\langle Z_r, \mathrm{Gram}(p_v : v \in V_r)\rangle = 0$ and $\mathrm{corank}\, Z_r = r$. Thus Theorem 11.1.2 applies and the claim follows.

Example 3: The family of graphs G_r. This family of graphs was considered in the study of the Colin de Verdière graph parameter [43]. For any integer $r \geq 2$ consider an equilateral triangle and subdivide each side into $r-1$ equal segments. Through these points draw line segments parallel to the sides of the triangle. This construction creates a triangulation of the big triangle into $(r-1)^2$ congruent equilateral triangles. The graph $G_r = (V_r, E_r)$ corresponds to the edge graph of this triangulation. Clearly, the graph G_r has $\binom{r+1}{2}$ vertices, which we denote (i,l) for $l \in [r]$ and $i \in [r-l+1]$. For any fixed $l \in [r]$ we say that the vertices $(1,l),\dots,(r-l+1,l)$ are at level l. Note that $G_2 = K_3 = F_2$, $G_3 = F_3$, but $G_r \neq F_r$ for $r \geq 4$. Fix an integer $r \geq 2$. We consider the following framework in $\mathbb{R}^r$ for the graph G_r:

$$p_{(i,1)} = e_i \ \forall i \in [r] \ \text{ and } p_{(i,l)} = p_{(i,l-1)} + p_{(i+1,l-1)} \ \forall l \geq 2 \text{ and } i \in [r-l+1]. \quad (11.3)$$

In Section 10.1.4 it is shown that for any $r \geq 2$ the partial G_r-partial matrix that corresponds to the framework defined in (11.3) has a unique psd completion. We now recover this result, using Theorem 11.1.2.

First we show that this framework satisfies (11.1). For this, consider a matrix $R \in \mathcal{S}^r$ such that $p_{(i,l)}^\mathsf{T} R\, p_{(i',l')} = 0$ for every $\{(i,l),(i',l')\} \in V_r \cup E_r$. Specializing this relation for $i' = i \in [r]$ and $l' = l = 1$ we get that $R_{ii} = 0$ for all $i \in [r]$ and for $i' = i+1$ and $l = l' = 1$ we get that $R_{i,i+1} = 0$ for $i \in [r-1]$. Similarly, for $i' = i+1$ and $l' = l \geq 2$ we get that $R_{i,i+l} = 0$ for all $i \in [r-l]$ and thus $R = 0$.

We call a triangle in G_r *black* if it is of the form $\{(i,l),(i+1,l),(i,l+1)\}$ and we denote by $\mathcal{B}_r$ the set of black triangles in G_r. The black triangles in G_5 are illustrated in Figure 10.2 as the shaded triangles. Let $Z_r = \sum_{t \in \mathcal{B}_r} u_t u_t^\mathsf{T}$ where the vector $u_t \in \mathbb{R}^{\binom{r+1}{2}}$ is defined as follows: If $t \in \mathcal{B}_r$ corresponds to the black triangle $\{(i,l),(i+1,l),(i,l+1)\}$ then $u_t(i,l) = u_t(i+1,l) = 1$, $u_t(i,l+1) = -1$ and 0 otherwise. Since $|\mathcal{B}_r| = \binom{r+1}{2} - r$ and the vectors $(u_t)_{t \in \mathcal{B}_r}$ are linearly independent we have that $\operatorname{corank} Z_r = r$. Moreover, as every edge of G_r belongs to exactly one black triangle we have that Z_r is supported by G_r. By construction of the framework we have that $\sum_{(i,l) \in V_r} p_{(i,l)} u_t = 0$ for all $t \in \mathcal{B}_r$ which implies that $\langle \mathrm{Gram}(p_{(i,l)} : (i,l) \in V_r), Z_r \rangle = 0$. Thus Theorem 11.1.2 applies and the claim follows.

Example 4: Tensor products of graphs. This construction was considered in [96], where universally rigid frameworks were used as a tool to construct uniquely colorable graphs. The original construction was carried out in the Euclidean setting for a suspension bar framework. Here we present the construction in the spherical setting which, as we will see in Section 11.2.3, is equivalent.

Let $H = ([n], E)$ be a k-regular graph satisfying $\max\limits_{2 \leq i \leq n} |\lambda_i| < k/(r-1)$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of its adjacency matrix A_H. For $r \in \mathbb{N}$ we let $G_r = (V_r, E_r)$ denote the graph $K_r \times H$, obtained by taking the tensor product of the complete graph K_r and the graph H. By construction, the adjacency matrix of G_r is the tensor product of the adjacency matrices of K_r and H: $A_{G_r} = A_{K_r} \otimes A_H$. Let us denote the vertices of G_r by the pairs (i,h) where $i \in [r]$ and $h \in V(H)$.

Let $w_1, \ldots, w_r \in \mathbb{R}^{r-1}$ be vectors that linearly span $\mathbb{R}^{r-1}$ and moreover satisfy $\sum_{i=1}^r w_i = 0$. We construct a framework for G_r in $\mathbb{R}^r$ by assigning to all nodes (i,h) for $h \in V(H)$ the vector $p_{(i,h)} = w_i$, for each $i \in [r]$. We now show, using Theorem 11.1.2, that the associated G_r-partial matrix admits a unique psd completion.

First we show that this framework satisfies (11.1). For this, consider a matrix $R \in \mathcal{S}^{r-1}$ satisfying $p_{(i,h)}^\mathsf{T} R p_{(i',h')} = 0$ for every $\{(i,h),(i',h')\} \in V_r \cup E_r$. This implies that $w_i^\mathsf{T} R w_j = 0$ for all $i,j \in [r]$ and as $\mathrm{lin}\{w_i w_j^\mathsf{T} + w_j w_i^\mathsf{T} : i,j \in [r]\} = \mathcal{S}^{r-1}$ it follows that $R = 0$.

Next consider the matrix $Z_k = I_{rn} + \frac{1}{k} A_{G_r} \in \mathcal{S}^{rn}$, where I_{rn} denotes the identity matrix of size rn. Notice that the matrix Z_r is by construction supported by G_r. One can verify directly that $\langle \mathrm{Gram}(p_{(i,h)} : (i,h) \in V_r), Z \rangle = 0$. The eigenvalues of A_{K_r} are $r-1$ with multiplicity one and -1 with multiplicity $r-1$. This fact combined with the assumption on the eigenvalues of H implies that Z_r is positive semidefinite with $\operatorname{corank} Z_r = r-1$. Thus Theorem 11.1.2 applies and the claim follows.

Example 5: The odd cycle C_5. The last example illustrates the fact that sometimes the sufficient conditions from Theorem 11.1.2 cannot be used to show existence of a unique psd completion. Here we consider the 5-cycle graph $G = C_5$ (although it is easy to generalize the example to arbitrary odd cycles).

First we consider the framework in $\mathbb{R}^2$ given by the vectors

$$p_i = (\cos(4(i-1)\pi/5), \sin(4(i-1)\pi/5))^{\mathsf{T}} \quad \text{for } 1 \le i \le 5.$$

The corresponding C_5-partial matrix has a unique psd completion, and this can be shown using Theorem 11.1.2.

It is easy to see that (11.1) holds. Let A_{C_5} denote the adjacency matrix of C_5 and recall that its eigenvalues are $2\cos\frac{2\pi}{5}$ and $-2\cos\frac{\pi}{5}$, both with multiplicity two and 2 with multiplicity one. Define $Z = 2\cos\frac{\pi}{5}I + A_{C_5}$ and notice that $Z \succeq 0$ and $\operatorname{corank} Z = 2$. Moreover, one can verify that $\sum_{j\in[5]} Z_{ij}p_j = 0$ for all $i \in [5]$ which implies that $\langle Z, \operatorname{Gram}(p_1,\ldots,p_5)\rangle = 0$. Thus Theorem 11.1.2 applies and the claim follows.

Next we consider another framework for C_5 in $\mathbb{R}^2$ given by the vectors

$$q_1 = (1,0)^{\mathsf{T}}, q_2 = (-1/\sqrt{2}, 1/\sqrt{2})^{\mathsf{T}}, q_3 = (0,-1)^{\mathsf{T}}, q_4 = (1/\sqrt{2}, 1/\sqrt{2})^{\mathsf{T}},$$
$$q_5 = (-1/\sqrt{2}, -1/\sqrt{2})^{\mathsf{T}}.$$

We now show that the corresponding C_5-partial matrix admits a unique psd completion. This cannot be shown using Theorem 11.1.2 since there does not exist a nonzero matrix $Z \in \mathscr{S}^5$ supported by C_5 satisfying $\langle Z, \operatorname{Gram}(q_1,\ldots q_5)\rangle = 0$. Nevertheless one can prove that there exists a unique psd completion by using the following geometric argument.

Let $X \in \mathscr{S}^5_+$ be a psd completion of the partial matrix and set $\vartheta_{ij} = \arccos X_{ij} \in [0,\pi]$ for $1 \le i \le j \le 5$. Then, $\vartheta_{12} = \vartheta_{23} = \vartheta_{34} = \vartheta_{15} = 3\pi/4$ and $\vartheta_{45} = \pi$. Therefore, the following linear equality holds:

$$\sum_{i=1}^{5} \vartheta_{i,i+1} = 4\pi \tag{11.4}$$

(where indices are taken modulo 5). As we will see this implies that the remaining angles are uniquely determined by the relations:

$$\vartheta_{i,i+2} + \vartheta_{i,i+1} + \vartheta_{i+1,i+2} = 2\pi \quad \text{for } 1 \le i \le 5 \tag{11.5}$$

and thus that X is uniquely determined. To see why the identities (11.5) hold, we use the well known fact that the angles ϑ_{ij} satisfy the triangle inequalities (recall Theorem 4.3.3):

$$\vartheta_{12} + \vartheta_{23} + \vartheta_{13} \le 2\pi,\ -\vartheta_{13} - \vartheta_{14} + \vartheta_{34} \le 0,\ \vartheta_{14} + \vartheta_{45} + \vartheta_{15} \le 2\pi. \tag{11.6}$$

Summing up the three inequalities in (11.6) and combining with (11.4), we deduce that equality holds throughout in (11.6). This permits to derive the values of $\vartheta_{13} = \pi/2$ and $\vartheta_{14} = \pi/4$ and we can proceed analogously for the remaining angles.

11.2 Universal rigidity of tensegrity frameworks

Our goal in this section is to give a concise and self-contained treatment of some known results concerning the universal rigidity of tensegrity frameworks. In particular, building on ideas from the two previous sections, we give a short and elementary proof of Connelly's sufficient condition for universal rigidity for both generic and non-generic tensegrity frameworks. Lastly, we also investigate the relation of our sufficient condition from Theorem 11.1.2 (for the Gram setting) to Connelly's sufficient condition from Theorem 11.2.4 (for the Euclidean distance setting).

11.2.1 Connelly's sufficient condition

A framework $G(\mathbf{p})$ is called *d-dimensional* if $p_1, \cdots, p_n \in \mathbb{R}^d$ and their affine span is $\mathbb{R}^d$. A d-dimensional framework is said to be in *general position* if every $d+1$ vectors are affinely independent. Given a framework $G(\mathbf{p})$ in $\mathbb{R}^d$, its *configuration matrix*, is the n-by-d matrix P whose rows are the vectors $p_1^\mathsf{T}, \dots, p_n^\mathsf{T}$, so that $PP^\mathsf{T} = \mathrm{Gram}(p_1, \dots, p_n)$. The framework $G(\mathbf{p})$ is said to be *generic* if the coordinates of the vectors $p_1, \dots, p_n$ are algebraically independent over the rational numbers.

11.2.1 Definition. *Let $G = ([n], E)$ be a tensegrity graph with $E = B \cup C \cup S$. A tensegrity framework $G(\mathbf{p})$ is said to* dominate *a tensegrity framework $G(\mathbf{q})$ if the following conditions hold:*

(i) $\|p_i - p_j\| = \|q_i - q_j\|$ *for all (bars)* $\{i, j\} \in B$,

(ii) $\|p_i - p_j\| \geq \|q_i - q_j\|$ *for all (cables)* $\{i, j\} \in C$,

(iii) $\|p_i - p_j\| \leq \|q_i - q_j\|$ *for all (struts)* $\{i, j\} \in S$.

Two frameworks $G(\mathbf{p})$ and $G(\mathbf{q})$ are called *congruent* if

$$\|p_i - p_j\| = \|q_i - q_j\|, \ \forall i \neq j \in [n].$$

Equivalently, this means that $G(\mathbf{q})$ can be obtained by $G(\mathbf{p})$ by a rigid motion of the Euclidean space. Our main focus in this section is on frameworks which, up to the group of rigid motions of the Euclidean space, admit a unique realization.

11.2.2 Definition. *A tensegrity framework $G(\mathbf{p})$ is called* universally rigid *if it is congruent to any tensegrity it dominates.*

An essential ingredient for characterizing universally rigid tensegrities is the notion of equilibrium stress matrix which we now introduce.

11.2.3 Definition. *A matrix $\Omega \in \mathscr{S}^n$ is called an* equilibrium stress matrix *for a tensegrity framework $G(\mathbf{p})$ if it satisfies:*

(i) $\Omega_{ij} = 0$ *for all* $\{i, j\} \in \overline{E}$.

(ii) $\Omega e = 0$ *and* $\Omega P = 0$, *i.e.,* $\sum_{j \in V} \Omega_{ij} p_j = 0$ *for all* $i \in V$.

(iii) $\Omega_{ij} \geq 0$ *for all (cables)* $\{i, j\} \in C$ *and* $\Omega_{ij} \leq 0$ *for all (struts)* $\{i, j\} \in S$.

Note that, by property (i) combined with the condition $\Omega e = 0$, any equilibrium stress matrix Ω can be written as $\Omega = \sum_{\{i,j\} \in E} \Omega_{ij} F_{ij}$, where $F_{ij} = (e_i - e_j)(e_i - e_j)^\mathsf{T}$.

The following result, due to R. Connelly, establishes a *sufficient condition* for determining the universal rigidity of tensegrity frameworks. All the ingredients for its proof are already present in [33], although there is no explicit statement of the theorem there. An exact formulation and a proof of Theorem 11.2.4 can be found in the (unpublished) work [34]. We now give an elementary proof of Theorem 11.2.4 which relies only on basic properties of positive semidefinite matrices. Our proof goes along the same lines as the proof of Theorem 11.1.2 and it is substantially shorter and simpler in comparison to Connelly's original proof.

11.2.4 Theorem. *Let $G = ([n], E)$ be a tensegrity graph with $E = B \cup C \cup S$ and let $G(\mathbf{p})$ be a tensegrity framework in $\mathbb{R}^d$ such that $p_1, \dots, p_n$ affinely span $\mathbb{R}^d$. Assume there exists an equilibrium stress matrix Ω for $G(\mathbf{p})$ such that:*

(i) *Ω is positive semidefinite.*

(ii) *Ω has corank $d+1$.*

(iii) *For any matrix $R \in \mathscr{S}^d$ the following holds:*

$$(p_i - p_j)^{\mathsf{T}} R\,(p_i - p_j) = 0 \;\forall \{i,j\} \in B \cup \{\{i,j\} \in C \cup S : \Omega_{ij} \neq 0\} \Longrightarrow R = 0. \tag{11.7}$$

Then, $G(\mathbf{p})$ is universally rigid.

Proof. Assume that $G(\mathbf{p})$ dominates another framework $G(\mathbf{q})$, our goal is to show that $G(\mathbf{p})$ and $G(\mathbf{q})$ are congruent. Recall that P is the $n \times d$ matrix with the vectors $p_1, \cdots, p_n$ as rows and define the augmented $n \times (d+1)$ matrix $P_a = \begin{pmatrix} P & e \end{pmatrix}$ obtained by adding the all-ones vector as last column to P. Set $X = PP^{\mathsf{T}}$ and $X_a = P_a P_a^{\mathsf{T}}$, so that $X_a = X + ee^{\mathsf{T}}$. As the tensegrity $G(\mathbf{p})$ is d-dimensional, we have that rank $X_a = d+1$. We claim that $\operatorname{Ker} X_a = \operatorname{Ran} \Omega$. Indeed, as Ω is an equilibrium stress matrix for $G(\mathbf{p})$, we have that $\Omega P_a = 0$ and thus $\Omega X_a = 0$. This implies that $\operatorname{Ran} X_a \subseteq \operatorname{Ker} \Omega$ and, as $\operatorname{corank} \Omega = d+1 = \operatorname{rank} X_a$, it follows that $\operatorname{Ker} X_a = \operatorname{Ran} \Omega$.

Let Y denote the Gram matrix of the vectors $q_1, \cdots, q_n$. We claim that $\operatorname{Ker} Y \supseteq \operatorname{Ker} X_a$. Indeed, we have that

$$0 \le \langle \Omega, Y \rangle = \langle \sum_{\{i,j\} \in E} \Omega_{ij} F_{ij}, Y \rangle \le \sum_{\{i,j\} \in E} \Omega_{ij} \langle F_{ij}, X_a \rangle = \langle \Omega, X_a \rangle = 0. \tag{11.8}$$

The first inequality follows from the fact that $\Omega, Y \succeq 0$; the second inequality holds since $\Omega_{ij} \langle F_{ij}, Y \rangle \le \Omega_{ij} \langle F_{ij}, X \rangle = \Omega_{ij} \langle F_{ij}, X_a \rangle$ for all edges $\{i,j\} \in E$, using the fact that $G(\mathbf{p})$ dominates $G(\mathbf{q})$ and the sign conditions on Ω. Therefore equality holds throughout in (11.8). This gives $\langle \Omega, Y \rangle = 0$, implying $Y\Omega = 0$ (since $Y, \Omega \succeq 0$) and thus $\operatorname{Ker} Y \supseteq \operatorname{Ran} \Omega = \operatorname{Ker} X_a$.

As $\operatorname{Ker} Y \supseteq \operatorname{Ker} X_a$, we deduce that $\operatorname{Ker}(Y - X_a) \supseteq \operatorname{Ker} X_a$ and thus $Y - X_a$ can be written as

$$Y - X_a = P_a R P_a^{\mathsf{T}} \;\text{ for some matrix }\; R = \begin{pmatrix} A & b \\ b^{\mathsf{T}} & c \end{pmatrix} \in \mathscr{S}^{d+1}, \tag{11.9}$$

where $A \in \mathscr{S}^d$, $b \in \mathbb{R}^d$ and $c \in \mathbb{R}$.

As equality holds throughout in (11.8) holds, we obtain $\Omega_{ij} \langle F_{ij}, Y - X_a \rangle = 0$ for all $\{i,j\} \in C \cup S$. Therefore, $\langle F_{ij}, P_a R P_a^{\mathsf{T}} \rangle = (p_i - p_j)^{\mathsf{T}} A (p_i - p_j) = 0$ for all $\{i,j\} \in B$ and for all $\{i,j\} \in C \cup S$ with $\Omega_{ij} \neq 0$. Using condition (iii), this implies that $A = 0$. Now, using (11.9) and the fact that $A = 0$, we obtain that

$$(Y - X_a)_{ij} = b^{\mathsf{T}} p_i + b^{\mathsf{T}} p_j + c \quad \text{for all } i,j \in [n].$$

From this follows that

$$\|q_i - q_j\|^2 = Y_{ii} + Y_{jj} - 2Y_{ij} = (X_a)_{ii} + (X_a)_{jj} - 2(X_a)_{ij} = \|p_i - p_j\|^2$$

for all $i,j \in [n]$, thus showing that $G(\mathbf{p})$ and $G(\mathbf{q})$ are congruent. □

Notice that the assumptions of the theorem imply that $n \ge d+1$. Moreover, for $n = d+1$ we get that Ω is the zero matrix in which case (11.7) is satisfied only for $G = K_n$ and $C = S = \emptyset$. Hence Theorem 11.2.4 is useful only in the case when $n \geq d + 2$.

There is a natural pair of primal and dual semidefinite programs attached to a given tensegrity framework $G(\mathbf{p})$:

$$\begin{aligned} \sup_X \{0 : X \succeq 0, \quad & \langle F_{ij}, X\rangle = \|p_i - p_j\|^2 \text{ for } \{i,j\} \in B, \\ & \langle F_{ij}, X\rangle \leq \|p_i - p_j\|^2 \text{ for } \{i,j\} \in C, \\ & \langle F_{ij}, X\rangle \geq \|p_i - p_j\|^2 \text{ for } \{i,j\} \in S\}, \end{aligned} \tag{11.10}$$

$$\begin{aligned} \inf_{y,Z} \{\textstyle\sum_{ij\in E} y_{ij}\|p_i - p_j\|^2 : \quad & Z = \textstyle\sum_{ij\in E} y_{ij}F_{ij} \succeq 0, \\ & y_{ij} \geq 0 \text{ for } \{i,j\} \in C, \\ & y_{ij} \leq 0 \text{ for } \{i,j\} \in S\}. \end{aligned} \tag{11.11}$$

The feasible (optimal) solutions of the primal program (11.10) correspond to the frameworks $G(\mathbf{q})$ that are dominated by $G(\mathbf{p})$, while the optimal solutions to the dual program (11.11) correspond to the positive semidefinite equilibrium stress matrices for the tensegrity framework $G(\mathbf{p})$.

Both matrices $X = PP^\mathsf{T}$ and $X_a = P_a P_a^\mathsf{T}$ (defined in the proof of Theorem 11.2.4) are primal optimal, with $\operatorname{rank} X = d$ and $\operatorname{rank} X_a = d + 1$. Hence, a psd equilibrium stress matrix Ω satisfies the conditions (i) and (ii) of Theorem 11.2.4 precisely when the pair (X_a, Ω) is a strict complementary pair of primal and dual optimal solutions.

In the case of bar frameworks (i.e., $C = S = \emptyset$), condition (iii) of Theorem 11.2.4 expresses the fact that the matrix $X = \operatorname{Gram}(p_1, \ldots, p_n)$ is an extreme point of the feasible region of (11.10). Moreover, X_a lies in its relative interior (since $\operatorname{Ker} Y \supseteq \operatorname{Ker} X_a$ for any primal feasible Y, as shown in the proof of Theorem 11.2.4)).

11.2.5 Remark. *In the terminology of Connelly, the condition (11.7) says that* the edge directions $p_i - p_j$ *of* $G(\mathbf{p})$ *for all edges* $\{i,j\} \in B$ *and all edges* $\{i,j\} \in C \cup S$ *with nonzero stress* $\Omega_{ij} \neq 0$ do not lie on a conic at infinity.

Observe that this condition cannot be omitted in Theorem 11.2.4. This is illustrated by the following example, taken from [3]. Consider the graph G on 4 nodes with edges $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ and $\{2,4\}$, and the 2-dimensional bar framework $G(p)$ given by

$$p_1 = (-1,0)^\mathsf{T},\ p_2 = (0,0)^\mathsf{T},\ p_3 = (1,0)^\mathsf{T} \textit{ and } p_4 = (0,1)^\mathsf{T}.$$

Clearly, the framework $G(p)$ is not universally rigid (as one can rotate p_4 and get a new framework, which is equivalent but not congruent to $G(\mathbf{p})$). On the other hand, the matrix $\Omega = (1,-2,1,0)(1,-2,1,0)^\mathsf{T}$ is the only equilibrium stress matrix for $G(\mathbf{p})$, it is positive semidefinite with corank 3. *Observe however that the condition (11.7) does not hold (since the nonzero matrix $R = e_1 e_2^\mathsf{T} + e_2 e_1^\mathsf{T}$ satisfies $(p_i - p_j)^\mathsf{T} R (p_i - p_j) = 0$ for all $\{i,j\} \in E$).*

11.2.2 Generic universally rigid frameworks

It is natural to ask for a converse of Theorem 11.2.4. This question has been settled recently in [54] in the affirmative for generic frameworks (cf. Theorem 11.2.9). First, we show that, for generic frameworks, the 'no conic at infinity' condition (11.7) can be omitted since it holds automatically. This result was obtained in [35]

(Proposition 4.3), but for the sake of completeness we have included a different and more explicit argument.

We need some notation. Given a framework $G(\mathbf{p})$ in $\mathbb{R}^k$, we let $\mathscr{P}_{\mathbf{p}}$ denote the $\binom{k+1}{2} \times |E|$ matrix, whose ij-th column contains the entries of the upper triangular part of the matrix $(p_i - p_j)(p_i - p_j)^\mathsf{T} \in \mathscr{S}^k$. For a subset $I \subseteq E$, $\mathscr{P}_{\mathbf{p}}(I)$ denotes the $\binom{k+1}{2} \times |I|$ submatrix of $\mathscr{P}_{\mathbf{p}}$ whose columns are indexed by edges in I.

11.2.6 Lemma. *Let $k \in \mathbb{N}$ and let $G = ([n], E)$ be a graph on $n \geq k+1$ nodes and with minimum degree at least k. Define the polynomial $\pi_{k,G}$ in kn variables by*

$$\pi_{k,G}(\mathbf{p}) = \sum_{I \subseteq E, |I| = \binom{k+1}{2}} (\det \mathscr{P}_{\mathbf{p}}(I))^2$$

for $\mathbf{p} = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^k$. Then, the polynomial $\pi_{k,G}$ has integer coefficients and it is not identically zero.

Proof. Notice that for the specific choice of parameters we have that $|E| \geq \frac{nk}{2} \geq \frac{(k+1)k}{2}$ and thus the sum in the definition of $\pi_{k,G}$ contains at least one term. It is clear that $\pi_{k,G}$ has integer coefficients. We show by induction on $k \geq 2$ that for every graph $G = ([n], E)$ with $n \geq k+1$ nodes and minimum degree at least k the polynomial $\pi_{k,G}$ is not identically zero.

For $k = 2$, we distinguish two cases: (i) $n = 3$ and (ii) $n \geq 4$. In case (i), $G = K_3$ and, for the vectors $p_1 = (0,0)^\mathsf{T}, p_2 = (1,0)^\mathsf{T}, p_3 = (0,1)^\mathsf{T}$, we have that $\pi_{2,G}(p) \neq 0$. In case (ii), we can now assume without loss of generality that the edge set contains the following subset $I = \{\{1,2\},\{1,3\},\{2,4\}\}$. For the vectors $p_1 = (0,0)^\mathsf{T}, p_2 = (1,0)^\mathsf{T}, p_3 = (0,1)^\mathsf{T}, p_4 = (2,1)^\mathsf{T}$, we have that $\det \mathscr{P}_{\mathbf{p}}(I) \neq 0$ and thus $\pi_{2,G}(\mathbf{p}) \neq 0$.

Let $k \geq 3$ and consider a graph $G = ([n], E)$ with $n \geq k+1$ and minimum degree at least k. Let $G \setminus n$ be the graph obtained from G by removing node n and all edges adjacent to it. Then, $G \setminus n$ has at least k nodes and minimum degree at least $k-1$. Hence, by the induction hypothesis, the polynomial $\pi_{k-1,G\setminus n}$ is not identically zero. Let $\mathbf{p} = \{p_1, \ldots, p_{n-1}\} \subseteq \mathbb{R}^{k-1}$ be a generic set of vectors and define $\tilde{\mathbf{p}} = \{\tilde{p}_1, \ldots, \tilde{p}_n\} \subseteq \mathbb{R}^k$, where $\tilde{p}_i = (p_i^\mathsf{T}, 0)^\mathsf{T} \in \mathbb{R}^k$ for $1 \leq i \leq n-1$ and $\tilde{p}_n = (\mathbf{0}, 1)^\mathsf{T} \in \mathbb{R}^k$. As $\mathbf{p}$ is generic, $\pi_{k-1,G\setminus n}(\mathbf{p}) \neq 0$ and thus $\det \mathscr{P}_{\mathbf{p}}(I) \neq 0$ for some subset $I \subseteq E(G \setminus n)$ with $|I| = \binom{k}{2}$. Say, node n is adjacent to the nodes $1, \ldots, k$ in G and define the edge subset $\tilde{I} = I \cup \{\{n,1\}, \ldots, \{n,k\}\} \subseteq E$. Then, the matrix $\mathscr{P}_{\tilde{\mathbf{p}}}(\tilde{I})$ has the block-form

$$\mathscr{P}_{\tilde{\mathbf{p}}}(\tilde{I}) = \begin{array}{c} \overbrace{\hphantom{\mathbf{0} \quad \ldots \quad \mathbf{0}}}^{\binom{k}{2}} \; \overbrace{\hphantom{-p_1 \quad \ldots \quad -p_k}}^{k} \\ \begin{pmatrix} & \mathscr{P}_{\mathbf{p}}(I) & & * & \ldots & * \\ \mathbf{0} & \ldots & \mathbf{0} & -p_1 & \ldots & -p_k \\ 0 & \ldots & 0 & 1 & \ldots & 1 \end{pmatrix} \end{array}.$$

As the vectors $p_1, \ldots, p_{n-1} \in \mathbb{R}^{k-1}$ were chosen to be generic, every k of them are affinely independent. This implies that the vectors $(-p_1^\mathsf{T}, 1)^\mathsf{T}, \ldots (-p_k^\mathsf{T}, 1)^\mathsf{T}$ are linearly independent. Hence, $\det \mathscr{P}_{\tilde{\mathbf{p}}}(\tilde{I}) \neq 0$ and thus $\pi_{k,G}(\tilde{\mathbf{p}}) \neq 0$. $\square$

11.2.7 Theorem. *[35] Let $G(\mathbf{p})$ be a generic d-dimensional framework and assume that G has minimum degree at least d. Then the edge directions of $G(\mathbf{p})$ do not lie on* *a conic at infinity; that is, the system* $\{(p_i - p_j)(p_i - p_j)^\mathsf{T} : \{i,j\} \in E\} \subseteq \mathscr{S}^d$ *has full rank* $\binom{d+1}{2}$.

Proof. As the framework $G(\mathbf{p})$ is d-dimensional, G must have at least $d+1$ nodes. By Lemma 11.2.6, the polynomial $\pi_{d,G}$ is not identically zero and thus, since $G(\mathbf{p})$ is generic, we have that $\pi_{d,G}(\mathbf{p}) \neq 0$. By definition of $\pi_{d,G}$ there exists $I \subseteq E$ with $|I| = \binom{d+1}{2}$ such that $\det \mathscr{P}_\mathbf{p}(I) \neq 0$. This implies that the system $\{(p_i - p_j)(p_i - p_j)^\mathsf{T} : \{i,j\} \in E\} \subseteq \mathscr{S}^d$ has full rank $\binom{d+1}{2}$. □

Next we show that for *generic* frameworks Theorem 11.2.4 remains valid even when (11.7) is omitted.

11.2.8 Corollary. *[35] Let* $G(\mathbf{p})$ *be a generic* d*-dimensional tensegrity framework. Assume that there exists a positive semidefinite equilibrium stress matrix* Ω *with corank* $d+1$. *Then* $G(\mathbf{p})$ *is universally rigid.*

Proof. Set $E_0 = \{\{i,j\} \in E : \Omega_{ij} \neq 0\}$ and define the subgraph $G_0 = ([n], E_0)$ of G. First we show that G_0 has minimum degree at least d. For this, we use the equilibrium conditions: For all $i \in [n]$, $\sum_{j:\{i,j\}\in E_0} \Omega_{ij} p_j = 0$, which give an affine dependency among the vectors p_i and p_j for $\{i,j\} \in E_0$. By assumption, $\mathbf{p}$ is generic and thus in general position, which implies that any $d+1$ of the vectors $p_1, \ldots, p_n$ are affinely independent. From this we deduce that each node $i \in [n]$ has degree at least d in G_0.

Hence we can apply Theorem 11.2.7 to the generic framework $G_0(\mathbf{p})$ and conclude that the system $\{(p_i - p_j)(p_i - p_j)^\mathsf{T} : \{i,j\} \in E_0\}$ has full rank $\binom{d+1}{2}$. This shows that the condition (11.7) holds. Now we can apply Theorem 11.2.4 to $G(\mathbf{p})$ and conclude that $G(\mathbf{p})$ is universally rigid. □

We note that for bar frameworks this fact has been also obtained independently by A. Alfakih using the related concepts of *dimensional rigidity* and *Gale matrices*. The notion of dimensional rigidity was introduced in [2] where a sufficient condition was obtained for showing that a framework is dimensionally rigid. In [4], using the concept of a Gale matrix, this condition was shown to be equivalent to the sufficient condition from Theorem 11.2.4 (for bar frameworks). Lastly, in [4] it is shown that for generic frameworks the notions of dimensional rigidity and universal rigidity coincide.

In the special case of bar frameworks, the converse of Corollary 11.2.8 was proved recently by Gortler and Thurston [54].

11.2.9 Theorem. *Let* $G(\mathbf{p})$ *be a generic* d*-dimensional bar framework and assume that it is universally rigid. Then there exists a positive semidefinite equilibrium stress matrix* Ω *for* $G(\mathbf{p})$ *with corank* $d+1$.

11.2.3 Connections with unique completability

In this section we investigate the links between the two notions of universally completable and universally rigid tensegrity frameworks. We start the discussion by defining the suspension of a tensegrity framework.

11.2.10 Definition. *Let* $G = (V = [n], E)$ *be a tensegrity graph with* $E = B \cup C \cup S$. *We denote by* $\nabla G = (V \cup \{0\}, E')$ *its* suspension tensegrity graph, *with* $E' = B' \cup C' \cup S'$

where $B' = B \cup \{\{0,i\} : i \in [n]\}$, $C' = S$ and $S' = C$. Given a tensegrity framework $G(\mathbf{p})$, we define the extended tensegrity framework $\nabla G(\widehat{\mathbf{p}})$ *where $\widehat{p}_i = p_i$ for all $i \in [n]$ and $\widehat{p}_0 = \mathbf{0}$.*

Our first observation is a correspondence between the universal completability of a tensegrity framework $G(\mathbf{p})$ and the universal rigidity of the extended tensegrity framework $\nabla G(\widehat{\mathbf{p}})$. The analogous observation in the setting of global rigidity was also made in [36] and [122].

11.2.11 Lemma. *Let $G(\mathbf{p})$ be a tensegrity framework and let $\nabla G(\widehat{\mathbf{p}})$ be its extended tensegrity framework as defined in Definition 11.2.10. Then, the tensegrity framework $G(\mathbf{p})$ is universally completable if and only if the extended tensegrity framework $\nabla G(\widehat{\mathbf{p}})$ is universally rigid.*

Proof. Notice that for any family of vectors $q_1, \ldots, q_n$, their Gram matrix satisfies the conditions:

$$\begin{aligned}
\langle E_{ij}, X\rangle &= p_i^{\mathsf{T}} p_j \quad \text{for all } \{i,j\} \in V \cup B,\\
\langle E_{ij}, X\rangle &\le p_i^{\mathsf{T}} p_j \quad \text{for all } \{i,j\} \in C,\\
\langle E_{ij}, X\rangle &\ge p_i^{\mathsf{T}} p_j \quad \text{for all } \{i,j\} \in S,
\end{aligned}$$

if and only if the Gram matrix of $q_0 = \mathbf{0}, q_1, \ldots, q_n$ satisfies:

$$\begin{aligned}
\langle F_{ij}, X\rangle &= \|p_i - p_j\|^2 \quad \text{for all } \{i,j\} \in B',\\
\langle F_{ij}, X\rangle &\le \|p_i - p_j\|^2 \quad \text{for all } \{i,j\} \in C',\\
\langle F_{ij}, X\rangle &\ge \|p_i - p_j\|^2 \quad \text{for all } \{i,j\} \in S',
\end{aligned}$$

which implies the claim. □

In view of Lemma 11.2.11 it is reasonable to ask whether Theorem 11.1.2 can be derived from Theorem 11.2.4 applied to the tensegrity framework $\nabla G(\widehat{\mathbf{p}})$. We will show that this is the case for bar frameworks, i.e., when $C = S = \emptyset$. Indeed, for a bar framework, the condition (11.7) from Theorem 11.2.4 applied to the suspension tensegrity framework $\nabla G(\widehat{\mathbf{p}})$ becomes

$$R \in \mathcal{S}^d,\ (p_i - p_j)^{\mathsf{T}} R\, (p_i - p_j) = 0 \text{ for all } \{i,j\} \in E \cup \{\{0,i\} : i \in [n]\} \Longrightarrow R = 0,$$

and, as $\widehat{\mathbf{p}}_0 = 0$, this coincides with the condition (11.1).

The following lemma shows that for bar frameworks there exists a one to one correspondence between equilibrium stress matrices for $\nabla G(\widehat{\mathbf{p}})$ and spherical stress matrices for $G(\mathbf{p})$. The crucial fact that we use here is that for bar frameworks there are no sign conditions for a spherical stress matrix for $G(\mathbf{p})$ or for an equilibrium stress matrix for $\nabla G(\widehat{\mathbf{p}})$.

11.2.12 Lemma. *Let $G(\mathbf{p})$ be a bar framework in $\mathbb{R}^d$ such that $p_1, \ldots, p_n$ span linearly $\mathbb{R}^d$. The following assertions are equivalent:*

(i) There exists an equilibrium stress matrix $\Omega \in \mathcal{S}_+^{n+1}$ for the framework $\nabla G(\widehat{\mathbf{p}})$ with $\operatorname{corank}\Omega = d + 1$.

(ii) There exists a spherical stress matrix for $G(\mathbf{p})$.

Proof. Let $P \in \mathbb{R}^{n\times d}$ be the configuration matrix of the framework $G(\mathbf{p})$ and let $\widehat{P}_a = \begin{pmatrix} \mathbf{0} & 1 \\ P & e \end{pmatrix}$. Write a matrix $\Omega \in \mathscr{S}_+^{n+1}$ in block-form as

$$\Omega = \begin{pmatrix} w_0 & w^\top \\ w & Z \end{pmatrix} \quad \text{where } Z \in \mathscr{S}_+^n, w \in \mathbb{R}^n, w_0 \in \mathbb{R}. \tag{11.12}$$

Notice that Ω is supported by ∇G precisely when Z is supported by G. The matrix Ω is a stress matrix for $\nabla G(\widehat{\mathbf{p}})$ if and only if $\Omega\widehat{P}_a = 0$ which is equivalent to

$$ZP = 0,\ w = -Ze,\ w_0 = -w^\top e. \tag{11.13}$$

Moreover, $\operatorname{Ker}\Omega = \operatorname{Ran}\widehat{P}_a$ if and only if $\operatorname{Ker}Z = \operatorname{Ran}P$, so that $\operatorname{corank}\Omega = d+1$ if and only if $\operatorname{corank}Z = d$. The lemma now follows easily: If Ω satisfies (i), then its principal submatrix Z satisfies (ii). Conversely, if Z satisfies (ii), then the matrix Ω defined by (11.12) and (11.13) satisfies (i). $\square$

Summarizing, we established that in the special case of bar frameworks $G(\mathbf{p})$ (i.e., $C = S = \emptyset$), Theorem 11.1.2 is equivalent to Theorem 11.2.4 applied to the extended bar framework. It is not clear whether this equivalence remains valid for arbitrary tensegrity frameworks. To deal with such frameworks, Lemma 11.2.12 has to be generalized to accommodate the sign conditions for the spherical stress matrix and the equilibrium stress matrix for $G(\mathbf{p})$ and $\nabla G(\widehat{\mathbf{p}})$, respectively.

Summary

This thesis revolves around the interplay between semidefinite programming and combinatorics. A semidefinite program is a convex program defined as the minimization of a linear function over an affine section of the cone of positive semidefinite matrices. A semidefinite program in canonical form looks as follows:

$$\begin{aligned} &\inf \langle A_0, X\rangle \\ &\text{subject to } \langle A_k, X\rangle = b_k, \qquad k = 1,\ldots,m \\ &X \succeq 0, \end{aligned} \tag{P}$$

where A_k $(0 \leq k \leq m)$ are n-by-n symmetric matrices and $\langle \cdot, \cdot \rangle$ denotes the usual Frobenius inner product of matrices.

The field of semidefinite programming has grown enormously in recent years and this success can be attributed to the fact that semidefinite programs have significant modeling power, exhibit powerful duality theory and there exist efficient algorithms, both in theory and in practice, for solving them. Starting with the seminal work of Goemans and Williamson on the max-cut problem, semidefinite programs have proven to be an invaluable tool in the design of approximation algorithms for hard combinatorial optimization problems.

The central question in this thesis is the search for *combinatorial conditions* guaranteeing the existence of low-rank optimal solutions to semidefinite programs. Results of this type are important for approximation algorithms since semidefinite programs are widely used as convex tractable relaxations for hard combinatorial problems. Then, rank one solutions typically correspond to optimal solutions of the initial discrete problem and low-rank optimal solutions can decrease the error of rounding methods and lead to improved performance guarantees.

Low-rank solutions to semidefinite programs are also relevant to the study of geometric representations of graphs. In this setting we consider representations obtained by assigning vectors to the vertices of a graph, where we impose some additional restrictions on the vectors labeling adjacent vertices (e.g. orthogonality, unit distance). Then, questions related to the existence of such representations in low dimension can be reformulated as the problem of deciding the existence of a low-rank solution to an appropriate semidefinite program and are connected to interesting graph properties.

In this thesis we focus on combinatorial conditions that capture the sparsity of the coefficient matrices of a semidefinite program. Specifically, with the semidefinite program (P), we associate a graph $\mathscr{A}_P = (V_P, E_P)$, called its *aggregate sparsity pattern*, where $V_P = \{1,\ldots,n\}$ and $ij \in E_P$ if and only if $(A_k)_{ij} \neq 0$ for some index $k \in \{0,1,\ldots,m\}$. Our main objective is to understand how to exploit the combinatorial structure of the aggregate sparsity pattern, to obtain guarantees for the existence of low-rank optimal solutions to semidefinite programs of the form (P).

Now, we give a brief description of the contents and the main contributions of this thesis. In Chapter 1 we motivate the study of the problem of identifying low-rank solutions to semidefinite programs. In Chapter 2 we present some background material concerning graph theory and positive semidefinite matrices. In Chapter 3 we collect several facts concerning the theory of semidefinite programming, a highlight being a link between semidefinite programming nondegeneracy and the Strong Arnold Property. Furthermore, in Chapter 4 we introduce and study the cut polytope and two of its most well-known relaxations.

In order to show that (P) has a low-rank optimal solution, we study a certain semidefinite program, known as the positive semidefinite matrix completion problem. In this setting, we are given as input a graph $G=(V,E)$ and a vector $a \in \mathbb{R}^{V\cup E}$ and the goal is to decide the *feasibility* of the following semidefinite program:

$$X_{ij} = a_{ij} \ (ij \in V \cup E), \ \ X \succeq 0. \tag{P_a}$$

In this thesis, we focus on the following variant of this problem, where we are given as input a vector $a \in \mathbb{R}^{V\cup E}$ for which (P_a) is feasible. Then, the goal is to exploit the combinatorial structure of G to show that (P_a) has a low-rank feasible solution.

In order to achieve this, in Chapter 5 we introduce a new graph parameter, called the Gram dimension of G, which we denote by $\text{gd}(G)$. Namely, $\text{gd}(G)$ is defined as the smallest integer $k \geq 1$ such that, for any $a \in \mathbb{R}^{V\cup E}$ for which (P_a) is feasible, it also has a feasible solution of rank at most k.

The first result derived in this chapter is that the graph parameter $\text{gd}(\cdot)$ is minor monotone, i.e., if H is a minor of G then $\text{gd}(H) \leq \text{gd}(G)$. Our main result is a complete characterization, in terms of minimal forbidden minors, of the graphs with small Gram dimension. Specifically, we show that

- $\text{gd}(G) \leq 2$ if and only if G has no K_3-minor,
- $\text{gd}(G) \leq 3$ if and only if G has no K_4-minor,
- $\text{gd}(G) \leq 4$ if and only if G has no K_5 and $K_{2,2,2}$-minors.

The relation of $\text{gd}(\cdot)$ with two other graph parameters that have been studied in the literature is investigated in Chapter 6. The first of these parameters deals with Euclidean realizations of graphs and the second with linear algebraic properties of positive semidefinite matrices, whose zero pattern is prescribed by a fixed graph. Our main result is that the characterization of graphs with $\text{gd}(\cdot) \leq 4$ implies the forbidden minor characterizations for both of these parameters.

In Chapter 7 we investigate complexity aspects of the low-rank positive semidefinite matrix completion problem. Specifically, we show that for any fixed integer $k \geq 2$, given a graph $G=(V,E)$ and a vector $a \in \mathbb{Q}^{V\cup E}$, it is NP-hard to decide whether (P_a) has a feasible solution of rank at most k.

Returning to the motivating question of identifying guarantees for the existence of low-rank optimal solutions to the semidefinite program (P), it is an easy consequence of the definition of the Gram dimension, that whenever (P) attains its optimum, it also has an optimal solution of rank at most $\text{gd}(\mathscr{A}_P)$.

In Chapter 10 we focus on a certain class of semidefinite programs, for which we can improve on the $\text{gd}(\mathscr{A}_P)$ bound. These are semidefinite programs with a simple set of constraints, namely requiring that every feasible matrix has all its diagonal entries equal to one. Specifically, for a graph $G = ([n], E)$ and a vector of edge-weights $w \in \mathbb{R}^E$, we consider semidefinite programs of the form:

$$\min \sum_{ij \in E} w_{ij} X_{ij} \text{ s.t. } X_{ii} = 1 \ (i \in [n]), \ X \succeq 0. \tag{P_w}$$

To study semidefinite programs of this form, we introduce a new graph parameter, called the extreme Gram dimension of a graph, which we denote by egd(G). Namely, egd(G) is defined as the smallest integer $k \geq 1$ such that, for any $w \in \mathbb{R}^E$, the semidefinite program (P_w) has an optimal solution of rank at most k.

By definition, for any $w \in \mathbb{R}^E$, program (P_w) has an optimal solution of rank at most egd$(\mathscr{A}_{P_w})$. Moreover, we show that for any graph G, egd$(G) \leq$ gd(G) and this inequality can be strict (for example, this is the case for the complete graph). Thus, in some cases, the egd$(\mathscr{A}_{P_w})$ bound can be better than the gd$(\mathscr{A}_{P_w})$ bound.

Furthermore, we show that the graph parameter egd$(\cdot)$ is minor monotone, i.e., if H is a minor of G then egd$(H) \leq$ egd(G). It is known that, egd$(G) \leq 1$ if and only if G is a forest. Our main result in Chapter 10 is a characterization, in terms of two forbidden minors, of the graphs satisfying egd$(G) \leq 2$.

In Chapter 11 we derive a sufficient condition that allows us to construct partial matrices, admitting a unique completion to a full positive semidefinite matrix. The construction of such matrices is a crucial ingredient for deriving lower bounds for the parameters gd$(\cdot)$ and egd$(\cdot)$. Lastly, we determine interesting connections with the theory of universally rigid graphs.

Bibliography

[1] J. Agler, J.W. Helton, S. McCullough, and L. Rodman. Positive semidefinite matrices with a given sparsity patern. *Linear Algebra Appl.*, 107:101–149, 1988.

[2] A.Y. Alfakih. On dimensional rigidity of bar-and-joint frameworks. *Discrete Appl. Math.*, 155:1244–1253, 2007.

[3] A.Y. Alfakih. On bar frameworks, stress matrices and semidefinite programming. *Math. Program. Ser. B*, 129(1):113–128, 2011.

[4] A.Y. Alfakih. On the universal rigidity of generic bar frameworks. *Contrib. Discrete Math.*, 5:7–17, 2012.

[5] A.Y. Alfakih, A. Khandani, and H. Wolkowicz. Solving Euclidean distance matrix completion problems via semidefinite programming. *Comput. Optim. Appl.*, 12:13–30, 1999.

[6] F. Alizadeh. Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM J. Optim.*, 5:13–51, 1993.

[7] F. Alizadeh, J.A. Haeberly, and M.L. Overton. Complementarity and nondegeneracy in semidefinite programming. *Math. Program. Ser. B*, 77(2):129–162, 1997.

[8] N. Alon, K. Makarychev, Y. Makarychev, and A. Naor. Quadratic forms on graphs. *Invent. Math.*, 163:486–493, 2005.

[9] N. Alon and A. Naor. Approximating the cut-norm via Grothendieck's inequality. In *Proceedings of the 36th ACM STOC*, pages 72–80, 2004.

[10] S. Arnborg, D.G. Corneil, and A. Proskurowski. Complexity of finding embeddings in a k-tree. *SIAM J. Alg. Disc. Meth.*, 8:277–284, 1987.

[11] S. Arnborg, D.G. Corneil, and A. Proskurowski. Forbidden minors characterization of partial 3-trees. *Disc. Math.*, 8(1):1–19, 1990.

[12] V.I. Arnold. On matrices depending on parameters. *Russian Mathematical Surveys*, 26:29–43, 1971.

[13] S. Arora, E. Berger, G. Kindler, M. Safra, and E. Hazan. On non-approximability for quadratic programs. In *Proceedings of the 46th IEEE FOCS*, pages 206–215, 2005.

[14] A. Avidor and U. Zwick. Rounding two and three dimensional solutions of the sdp relaxation of max cut. In C. Chekuri et al., editor, *APPROX and RANDOM 2005*, LNCS, vol. 3624, pages 14–25, 2005.

[15] D. Avis and M. Deza. The cut cone, L1 embeddability, complexity, and multicommodity flows. *Networks*, 21(6):595–617, 1991.

[16] V. Picciali, A.Y. Alfakih, M.F. Anjos, and H. Wolkowicz. Euclidean distance matrices, semidefinite programming and sensor network localization. *Portugaliae Mathematica*, 68(1), 2011.

[17] F. Barahona. The max-cut problem on graphs not contractible to K_5. *Oper. Res. Lett.*, 2(3):107–111, 1983.

[18] F. Barahona. On cuts and matchings in planar graphs. *Math. Program.*, 60:53–68, 1993.

[19] F. Barahona and A. Mahjoub. On the cut polytope. *Math. Program.*, 36:157–173, 1986.

[20] W.W. Barrett, C.R. Johnson, and P. Tarazaga. The real positive definite completion problem for a simple cycle. *Linear Algebra Appl.*, 192:3–31, 1993.

[21] W.W. Barrett, C.R. Johnson, and P. Tarazaga. The real positive definite completion problem: cycle completability. *Memoirs Am. Math. Soc.*, 584, 1996.

[22] A. Barvinok. Problems of distance geometry and convex properties of quadratic maps. *Disc. Comput. Geom.*, 13:189–202, 1995.

[23] A. Barvinok. A remark on the rank of positive semidefinite matrices subject to affine constraints. *Disc. Comp. Geom.*, 25(1):23–31, 2001.

[24] A. Barvinok. *A Course in Convexity*, vol. 54. AMS, 2002.

[25] M. Belk. Realizability of graphs in three dimensions. *Disc. Comput. Geom.*, 37:139–162, 2007.

[26] M. Belk and R. Connelly. Realizability of graphs. *Disc. Comput. Geom.*, 37:125–137, 2007.

[27] G. Blekherman, P.A. Parrilo, and R. Thomas, editors. *Semidefinite Optimization and Convex Algebraic Geometry*. SIAM, 2012.

[28] H.L. Boedlander. A linear-time algorithm for finding tree-decompositions of small treewidth. *SIAM J. Comput.*, 25(6):1305–1317, 1993.

[29] M. Braverman, K. Makarychev, Y. Makarychev, and A. Naor. The Grothendieck constant is strictly smaller than Krivine's bound. In *Proceedings of the 52nd IEEE FOCS*, pages 453-462, 2011.

[30] J. Briët. *Grothendieck Inequalities, Nonlocal Games and Optimization*. PhD thesis, University of Amsterdam, 2012.

[31] J. Briët, F.M. de Oliveira Filho and F. Vallentin. Grothendieck inequalities for semidefinite programs with rank constraint. arxiv:1011.1754, 2010.

[32] R. Cleve, P. Hoyer, B. Toner, and J. Watrous. Consequences of limits of nonlocal strategies. In *Proceedings of the 19th IEEE CCC*, pages 236–249, 2004.

[33] R. Connelly. Rigidity and energy. *Invent. Math.*, 66(1):11–33, 1982.

[34] R. Connelly. Stress and stability. Available at `http://www.math.cornell.edu/~connelly/Tensegrity.Chapter.2.ps1,2001.`

[35] R. Connelly. Generic global rigidity. *Disc. Comput. Geom.*, 33:549–563, 2005.

[36] R. Connelly and W.J. Whiteley. Global rigidity: The effect of coning. *Disc. Comput. Geom.*, 43(4):717–735, 2010.

[37] G.M. Crippen and T.F. Havel. *Distance Geometry and Molecular Conformation*. John Wiley & Sons, 1988.

[38] A.M. Davie. Lower bound for K_G. Unpublished note, 1984.

[39] E. de Klerk. *Aspects of Semidefinite Programming - Interior Point Algorithms and Selected Applications*. Kluwer, 2002.

[40] W.F. de la Vega and C. Kenyon-Mathieu. Linear programming relaxations of maxcut. In *Proceedings of the 18th SODA*, pages 53–61, 2007.

[41] J. de Leeuw and W. Heiser. Theory of multidimensional scaling. In P.H. Krishnaiah and L.N. Kanal, editors, *Handbook of Statistics*, vol. 2, pages 285–316, Elsevier, 1982.

[42] Y. Colin de Verdière. Sur un nouvel invariant des graphes et un critère de planarité. *J. Comb. Theory Ser. B*, 50:11–21, 1990.

[43] Y. Colin de Verdière. Multiplicities of eigenvalues and tree-width of graphs. *J. Comb. Theory Ser. B*, 74(2):121–146, 1998.

[44] M. Deza and M. Laurent. *Geometry of Cuts and Metrics*. Springer, 1997.

[45] R.J. Duffin. Topology of series-parallel networks. *J. Math. Annal. Appl.*, 10(2):303–313, 1965.

[46] M. E.-Nagy, M. Laurent, and A. Varvitsiotis. On bounded rank positive semidefinite matrix completions of extreme partial correlation matrices. arXiv:1205.2040, 2012.

[47] M. E.-Nagy, M. Laurent, and A. Varvitsiotis. Complexity of the positive semidefinite matrix completion problem with a rank constraint. In K. Bezdek, A. Deza, and Y. Ye, editors, *Discrete Geometry and Optimization*, vol. 69 of Fields Institute Communications, pages 105–120, Springer, 2013.

[48] P.C. Fishburn and J.A. Reeds. Bell inequalities, Grothendieck's constant, and root two. *SIAM J. Disc. Math.*, 7(1):48–56, 1994.

[49] S. Friedland and R. Loewy. Subspaces of symmetric matrices containing matrices with a multiple first eigenvalue. *Pacific Journal of Mathematics*, 62(22):389–399, 1976.

[50] C. Godsil. The Colin de Verdière Number. Unpublished note. Available at `http://quoll.uwaterloo.ca/mine/Notes/cdv.pdf.`

[51] M.X. Goemans and D.P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. ACM*, 42:1115–1145, 1995.

[52] F. Göring, C. Helmberg, and S. Reiss. Graph realizations associated with minimizing the maximum eigenvalue of the Laplacian. *Math. Program.*, 131(1–2):95–11, 2012.

[53] F. Göring, C. Helmberg, and M. Wappler. Embedded in the shadow of the separator. *SIAM J. Optim.*, 19(1):472–501, 2008.

[54] S.J. Gortler and D. Thurston. Characterizing the universal rigidity of generic frameworks. arXiv:1001.0172, 2009.

[55] R. Grone, C.R. Johnson, E.M. Sá, and H. Wolkowicz. Positive definite completions of partial Hermitian matrices. *Linear Algebra Appl.*, 58:95–11, 1984.

[56] R. Grone, S. Pierce, and W. Watkins. Extremal correlation matrices. *Linear Algebra Appl.*, 134:63–70, 1990.

[57] A. Grothendieck. Résumé de la théorie métrique des produits tensoriels topologiques. *Bol. Soc. Mat. Sao Paolo.*, 8:1–79, 1953.

[58] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer, 1988.

[59] J. Håstad. Some optimal inapproximability results. *J. ACM*, 48:798–869, 2001.

[60] G. Haynes, C. Park, A. Schaeffer, J. Webster, and L.M. Mitchell. Orthogonal vector coloring. *Electron. J. Comb.*, 17(R55), 2010.

[61] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.

[62] D.K. Goldenberg, A.S. Morse, W. Whiteley, Y.R. Yang, B.D.O. Anderson, P.N. Belhumeur, J. Aspnes, and T. Eren. A theory of network localization. *IEEE T. Mobile Comput.*, 5(12):1663–1678, 2006.

[63] R. Karp. *Reducibility among combinatorial problems*, Springer US, 1972.

[64] R. Karp and C.H. Papadimitriou. On linear characterizations of combinatorial optimization problems. *SIAM J. Comput.*, 11(4):620–632, 1982.

[65] S. Khot, G. Kindler, E. Mossel, and R. O'Donnell. Optimal inapproximability results for MAX-CUT and other 2-variable CSPs? *SIAM J. Comput.*, 37(1):319–357, 2007.

[66] S. Khot and A. Naor. Sharp kernel clustering algorithms and their associated Grothendieck inequalities. In *Proceedings of the 21st ACM-SIAM SODA*, pages 664–683, 2010.

[67] S. Khot and A. Naor. Grothendieck-type inequalities in combinatorial optimization. *Commun. Pur. Appl. Math.*, 65(7):992–1035, 2012.

[68] T. Kloks. *Treewidth, Computations and Approximations*, LNCS, vol. 842, Springer, 1994.

[69] S. Kochen and E.P. Specker. The problem of hidden variables in quantum mechanics. *J. Math. Mech.*, 15:59–87, 1967.

[70] A. Kotlov. Spectral characterization of tree-width-two graphs. *Combinatorica*, 20(1):147–152, 2000.

[71] N. Krislock and H. Wolkowicz. Euclidean distance matrices and applications. In M.F. Anjos and J.B. Lasserre, editors, *Handbook on Semidefinite, Conic and Polynomial Optimization*, pages 879–914, Springer, 2012.

[72] J. Krivine. Sur la constante de Grothendieck. *C. R. Acad. Sci. Paris Sér. A-B,*, 284(8):445–446, 1997.

[73] K. Kuratowski. Sur le problème des courbes gauches en topologie. *Fund. Math.*, 15:271–283, 1930.

[74] M. Laurent. The real positive semidefinite completion problem for series-parallel graphs. *Linear Algebra Appl.*, 252:347–366, 1997.

[75] M. Laurent. A tour d'horizon on positive semidefinite and Euclidean distance matrix completion problems. In P.M. Pardalos and H. Wolkowicz, editors, *Topics in Semidefinite and Interior point methods*, vol. 18 of Fields Institute Communications, pages 51–76, 1998.

[76] M. Laurent. Polynomial instances of the positive semidefinite and Euclidean distance matrix completion problems. *SIAM J. Matrix Anal. A.*, 22:874–894, 2000.

[77] M. Laurent. Matrix completion problems, In C.A. Floudas and P.M. Pardalos, editors, *The Encyclopedia of Optimization*, vol. 3, pages 221–229, Kluwer, 2001.

[78] M. Laurent. Semidefinite relaxations for Max-Cut. In M. Grötschel, editor, *The sharpest cut: The Impact of Manfred Padberg and his work*, vol. 4 of MPS-SIAM Series in Optimization, pages 257–290, 2004.

[79] M. Laurent. Sums of squares, moment matrices and optimization over polynomials. In M. Putinar and S. Sullivant, editors, *Emerging Applications of Algebraic Geometry*, vol. 149 of IMA Volumes in Mathematics and its Applications, pages 157–270, Springer, 2009.

[80] M. Laurent and A. Varvitsiotis. Positive semidefinite matrix completion, universal rigidity and the Strong Arnold Property. arXiv:1301.6616, 2013.

[81] M. Laurent and A. Varvitsiotis. Computing the Grothendieck constant of some graph classes. *Oper. Res. Lett.*, 39(6):452–456, 2011.

[82] M. Laurent and A. Varvitsiotis. The Gram dimension of a graph. In A.R. Mahjoub et al., editors, In *Proceedings of the 2nd ISCO*, LNCS, vol. 7422, pages 356–367, 2012.

[83] M. Laurent and A. Varvitsiotis. A new graph parameter related to bounded rank positive semidefinite matrix completions. *Math. Program.*, Published online: 13 February 2013.

[84] C.-K. Li and B.-S. Tam. A note on extreme correlation matrices. *SIAM J. Matrix Anal. A.*, 15(3):903–908, 1994.

[85] J. Lindestrauss and A. Pelczynski. Absolutely summing operators in L_p-spaces and their applications. *Studia Math.*, 29:275–326, 1968.

[86] L. Lovász. Semidefinite programs and combinatorial optimization. Available at `http://www.cs.elte.hu/~lovasz/semidef.ps`.

[87] L. Lovász. On the Shannon capacity of graphs. *IEEE Trans. on Information Theory*, 25:1–7, 1979.

[88] L. Lovász and A. Schrijver. A Borsuk theorem for antipodal links and a spectral characterization of linklessly embedabble graphs. *P. Am. Math. Soc.*, 126(5):1275–1285, 1998.

[89] L. Lovász and A. Schrijver. On the null space of a Colin de Verdière matrix. *Annales de l' institut Fourier*, 49(3):1017–1026, 1999.

[90] L. Lovász and K. Vesztergombi. Geometric representations of graphs. In *Paul Erdös and his Mathematics*, Bolyai Soc. Math. Stud., 2002.

[91] M. Charikar, K. Makarychev, and Y. Makarychev. Near-optimal algorithms for unique games. In *Proceedings of the 38th ACM STOC*, pages 205-214, 2006.

[92] B. Gärtner and J. Matousek. *Approximation Algorithms and Semidefinite Programming*. Springer, 2012.

[93] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., 1979.

[94] A. Naor and G. Schechtman. An approximation scheme for quadratic form maximization on convex bodies. Manuscript, 2009.

[95] Y. Nesterov and A. Nemirovski. *Interior-Point Polynomial Algorithms in Convex Programming*. SIAM Studies in Applied Mathematics, 1994.

[96] I. Pak and D. Vilenchik. Constructing uniquely realizable graphs. Available at `http://www.math.ucla.edu/~pak/papers/GRP4.pdf1`.

[97] P.A. Parrilo and S. Lall. Semidefinite programming relaxations and algebraic optimization in control. *Eur. J. Contol*, 9(2–3):307–321, 2003.

[98] G. Pataki. Cone LP's and semidefinite programs: Geometry and a simplex type method. In *Integer programming and combinatorial optimization*, pages 162–174. Springer, 1996.

[99] G. Pataki. The geometry of semidefinite programming. In L. Vandenberghe, H. Wolkowicz, and R. Saigal, editors, *Handbook of Semidefinite Programming*, pages 29–65, Kluwer, 2000.

[100] R. Peeters. *Ranks and Structure of Graphs*. PhD thesis, University of Tilburg, 1995.

[101] R. Peeters. Orthogonal representations over finite fields and the chromatic number of graphs. *Combinatorica*, 16(3):417–431, 1996.

[102] A. Peres. Two simple proofs of the Kochen-Specker theorem. *Journal of Physics A*, 24:175–178, 1991.

[103] G. Pisier. Grothendieck's theorem for noncommutative C^*-algebras, with an appendix on Grothndieck's constants. *J. Funct. Anal.*, 29(3):397–415, 1978.

[104] G. Pisier. Grothendieck's theorem, past and present. *Bull. Amer. Math. Soc.*, 49:237–323, 2012.

[105] I. Pitowsky. New Bell inequalities for the singlet state: Going beyond the Grothendieck bound. *J. Math. Phys.*, 49:0121101, 2008.

[106] C. Delorme and S. Poljak. Combinatorial properties and the complexity of a max-cut approximation. *Eur. J. Combin.*, 14(313–333), 1993.

[107] C. Delorme and S. Poljak. The performance of an eigenvalue bound on the max-cut problem in some classes of graphs. *Discrete Math.*, 111(144–156), 1993.

[108] S. Poljak and Z. Tuza. On the expected relative error of the polyhedral approximation of the max-cut. *Oper. Res. Lett.*, 16:191–198, 1994.

[109] L. Khachiyan and L. Porkolab. On the complexity of semidefinite programs. *J. Global Optim.*, 10:351–365, 1997.

[110] M.V. Ramana. An exact duality theory for semidefinite programming and its complexity implications. *Math. Program.*, 77(1):129–162, 1997.

[111] P.M. Pardalos and M.V. Ramana. Semidefinite programming. In T. Terlaky, editor, *Interior point methods of Mathematical programming*. Kluwer, 1996.

[112] B. Recht, M. Fazel, and P.A. Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM Rev.*, 52(3):471–501, 2010.

[113] J.A. Reeds. A new lower bound on the real Grothendieck constant. Manuscript, 1991.

[114] N. Robertson and P.D. Seymour. Graph minors. XIII. The disjoint paths problem. *J. Comb. Theory Ser. B*, 63(1):65–110, 1995.

[115] N. Robertson and P.D. Seymour. Graph minors. XX. Wagners conjecture. *J. Comb. Theory Ser. B*, 92(2):325–357, 2004.

[116] N. Robertson, P.D. Seymour, and R. Thomas. A survey of linkless embeddings. In N. Robertson and P.D. Seymour, editors, *Graph Structure Theory*, pages 125–136, 1993.

[117] F.V. Saliola and W.J. Whiteley. Some notes on the equivalence of first-order rigidity in various geometries. arXiv:0709.3354v1, 2007.

[118] J.B. Saxe. Embeddability of weighted graphs in k-space is strongly NP-hard. In *Proceedings of the 17-th Allerton Conf. Comm. Control Comp.*, pages 480–489, 1979.

[119] G. Schoenebeck, L. Trevisan, and M. Tulsiani. Lovász-Schrijver LP relaxations of vertex cover and max cut. In *Proceedings of the 39th ACM STOC*, pages 302–310, 2007.

[120] A. Schrijver. *Theory of Linear and Integer Programming*. Wiley, 1986.

[121] A. Shapiro and M.K. Fan. On eigenvalue optimization. *SIAM J. Optim.*, 5(3):552–569, 1995.

[122] A. Singer and M. Cucuringu. Uniqueness of low-rank matrix completion by rigidity theory. *SIAM J. Matrix Anal. A.*, 31:1621–1641, 2010.

[123] A. Man-Cho So and Y. Ye. A semidefinite programming approach to tensegrity theory and realizability of graphs. In *Proceedings of the 17th ACM-SIAM SODA*, pages 766–775, 2006.

[124] H.E. Stanley. Spherical model as the limit of infinite spin dimensionality. *Phys. Rev.*, 176:718–721, 1968.

[125] L. Stockmeyer. Planar 3-colorability is polynomial complete. *SIGACT News*, 5(3):19–25, 1973.

[126] H. van der Holst. *Topological and Spectral Graph Characterizations*. PhD thesis, University of Amsterdam, 1996.

[127] H. van der Holst. Graphs with magnetic Schrödinger operators of low corank. *J. Comb. Theory Ser. B*, 84:311–339, 2002.

[128] H. van der Holst. On the "Largeur d'Arborescence". *J. Graph Theory*, 41:24–52, 2002.

[129] H. van der Holst. Two tree-width-like graph invariants. *Combinatorica*, 23(4):633–651, 2003.

[130] K. Wagner. Über eine Eigenschaft der Ebenen Komplexe. *Math. Ann.*, 114(1):570–590, 1937.

[131] S. Wehner. Tsirelson bounds for generalized Clauser-Horne-Shimony-Holt inequalities. *Phys. Rev. A*, 73:022110, 2006.

Index

List of Symbols

K_n	The complete graph on n nodes, page 21
C_n	The circuit graph on n nodes, page 21
$\mathscr{S}^n$	The set of n-by-n symmetric matrices, page 24
$\mathscr{S}^n_+$	The set of n-by-n positive semidefinite matrices, page 24
$\mathscr{S}^n_{++}$	The set of n-by-n positive definite matrices, page 24
$\mathscr{E}_n$	The set of n-by-n correlation matrices, page 33
$\mathscr{E}_{n,k}$	The rank-constrained elliptope, page 81
$\mathrm{rank} A$	The rank of a matrix, page 24
$\mathrm{corank} A$	The corank of a matrix, page 24
$\mathrm{tw}(\mathrm{G})$	The treewidth of a graph, page 23
$\mathrm{la}_{\boxtimes}(G)$	The strong largeur d'arborescence, page 109
$\mathrm{la}_{\square}(G)$	Largeur d'arborescence, page 23
$\vartheta(G)$	The Lovász theta function of the graph G, page 2
$\omega(G)$	The clique number of the graph G, page 2
$\nu^{=}(G)$	Graph parameter, page 77
$\mathrm{ed}(G)$	The Euclidean dimension of the graph G, page 75
$\kappa(r,G)$	The rank-r Grothendieck constant of the graph G, page 93
$\kappa(G)$	The Grothendieck constant of the graph G, page 93
$\mathrm{egd}(G)$	The extreme Gram dimension of the graph G, page 107
$\mathrm{gd}(G,a)$	The Gram dimension of the partial matrix $a \in \mathbb{R}^{V \cup E}$, page 53
$\mathrm{gd}(G)$	The Gram dimension of the graph G, page 52
$\mathscr{S}_+(G)$	The set of G-partial psd matrices that are completable to a positive semidefinite matrix, page 52
$\mathscr{S}_{++}(G)$	The set of G-partial psd matrices that are completable to a positive definite matrix, page 52
$\mathscr{E}(G)$	The elliptope of a graph G, page 47

$\mathscr{E}_k(G)$	The rank-constrained elliptope of the graph G, page 82
$\mathrm{MET}^{01}(G)$	The metric polytope of the graph G in $0,1$ variables, page 45
$\mathrm{CUT}^{\pm 1}(G)$	The cut polytope of the graph G in ± 1 variables, page 44
$\mathrm{CUT}^{01}(G)$	The cut polytope of the graph G in $0,1$ variables, page 43
$\delta_G(S)$	The cut vector defined by S, page 43
$G \Box G'$	Cartesian product of graphs, page 22
$G \times G'$	Tensor product of graphs, page 22
$G \boxtimes G'$	Strong product of graphs, page 22
$[n]$	The set $\{1, \ldots, n\}$, page 21
$\langle \cdot, \cdot \rangle$	The trace inner product of matrices, page 25
$\mathbb{S}^n$	The n-dimensional unit sphere, page 11
$\mathscr{T}_X$	Tangent space at X, page 35
∇G	The suspension of the graph G, page 76
$\mathrm{Pert}_C(x)$	The set of perturbations of the point x with respect to C, page 20
π_E	The projection operator on the edge set of the graph G , page 46
$F_C(x)$	The smallest face of the set C that contains x, page 20
$G(\mathbf{p})$	Tensegrity framework, page 130
K_G	Grothendieck constant, page 92
$L_{G,w}$	The Laplacian matrix of the edge-weighted graph (G, w), page 96
$O(d)$	The set of d-by-d orthogonal matrices, page 26
$\mathrm{Gram}(p_1, \ldots, p_n)$	The Gram marix of vectors $p_1, \ldots, p_n$., page 24